FUNDAMENTALS OF MATHEMATICS

VOL. I

R. P. GOEL

PITAMBAR PUBLISHING COMPANY

Based on the latest syllabus prescribed by the Central Board of Secondary Education, New Delhi under the New Education Policy.

FUNDAMENTALS OF MATHEMATICS

VOL. I

(FOR CLASS IX)

RAJENDRA PRASAD GOEL

Formerly, Senior Mathematics Teacher,
St. Xavier's School,
Delhi.

PITAMBAR PUBLISHING COMPANY
EDUCATIONAL PUBLISHERS
888, EAST PARK ROAD, KAROL BAGH,
NEW DELHI-110 005 (INDIA).

Published by :
PITAMBAR PUBLISHING COMPANY
Educational Publishers
888, East Park Road, Karol Bagh,
NEW DELHI-110 005 (INDIA)
Telephones : Off. : 526933, 776058, 770067
Resi. : 586788, 5721321, 5715182

First Edition : 1989
Reprinted : 1990

© Copyright Reserved

Price : Rs. 40/-

Code No. : 27265

Acc No - 15154

Printed at :
Piyush Printers Publishers Pvt. Ltd.
G - 12, Udyog Nagar,
Rohtak Road Industrial Area, New Delhi - 110041.
Telephone : 5472440

PREFACE

This book has been written for the students of Class IX of Secondary Schools affiliated to the Central Board of Secondary Education (CBSE), New Delhi, according to new syllabus developed by National Council of Educational Research and Training (NCERT), New Delhi, under the National Policy of Education.

In fact, it is an effort made to cater to the needs of the average students and the brighter students as well who have to learn Mathematics as a compulsory subject. Learning of Mathematics is essential for them to study other subjects as an integrated course and to face multi-dimensional and challenging situations in days to come. To make learning of the subject interesting and meaningful to the day-to-day needs of would be citizens, I have presented the contents in a new format which is based on my long experience of teaching the subject and varied writing experience of text-books on Mathematics.

The following are the special features of the book :

1. The emphasis has been laid on the clear understanding of basic concepts and principles by the students.
2. Treatment of the subject matter is simple and logical. The matter has been developed successively with the help of previous knowledge of the students.
3. Each chapter comprises several functional units and each unit is designed to cover one or two teaching periods.
4. Numerous worked-out examples have been provided for better understanding of the contents by the students and to explain the methodical working of the problems fully.
5. There are sufficient well graded questions related to our daily life in each exercise. Each exercise has been divided into three Sections A, B and C. Section A consists of objective or short-answer questions, while Section B has simple applications and Section C has more difficult questions.
6. After each chapter a review exercise has been added, enriched by the inclusion of questions set in public examinations for effective revision side by side.
7. Working rules, important results and points worth remembering and deductions are given at appropriate places.
8. Wherever necessary hints have been provided to help students to solve difficult questions with confidence.

9. The figures, diagrams and graphs given in the book are complete, neatly drawn and self-explanatory.

10. Only agreed conventions, popular symbols or notations and metric units have been used throughout the book.

11. Answers to questions have been carefully checked for their correctness and accuracy.

I am indebted to Dr. Vijay Bhushan Aggarwal, Founder Head of the Deptt. of Computer Science, Delhi University for writing Chapter 19 on Computer in the book.

I would like to request my fellow teachers and pupils to send their constructive criticism so that the book may be improved in its future editions.

RAJENDRA PRASAD GOEL

Ashok Vihar, Delhi
6th April, 1989

CONTENTS

Chapter *Page*

1. SETS **1—25**

1.1 Sets, 1.2 Cardinal Number, 1.3 Finite and Infinite Sets, 1.4 Equal Sets, 1.5 Equivalent Sets, 1.6 The Empty Set, 1.7 Subsets, 1.8 Dispoint Sets, 1.9 Universal Set, 1.10 Complement of a Set, 1.11 Union of Sets, 1.12 Intersection of Sets, 1.13 Properties of Operations with Sets, 1.14 Venn Diagram 1.15 Applications.

2. NUMBER SYSTEMS **26—41**

2.1 Natural Numbers, 2.2 Integers, 2.3 Rational Numbers, 2.4 Irrational Numbers, 2.5 Real Numbers.

3. SURDS **42—55**

3.1 Surd, 3.2 Order of Surds, 3.3 Pure and Mixed Surds, 3.4 Similar Surds, 3.5 Comparison of Surds, 3.6 Simple and Compound Surds, 3.7 Addition and Subtraction of Surds, 3.8 Multiplication of Surds, 3.9 Division of Surds, 3.10 Rationalisation of Surds, 3.11 Rationalisation of Monomial Surds, 3.12 Rationalisation of Binomial Quadratic Surds.

4. FUNCTIONS **56—66**

4.1 Graphs, 4.2 Rectangular Coordinate Systems, 4.3 Functions, 4.4 Graph of Functions.

5. POLYNOMIALS **67—80**

5.1 Polynomials, 5.2 Addition of Polynomials, 5.3 Subtraction of Polynomials, 5.4 Multiplication of Polynomials, 5.5 Division of Polynomials, 5.6 Properties of Polynomials, 5 7 Remainder Theorem.

6. FACTORISING POLYNOMIALS **81—93**

6.1 Factors, 6.2 Factorising Quadratic Polynomials, 6.3 Factorising Difference of Two Squares, 6.4 Factorising Trinomials, 6.5 Factorising Trinomials, 6.6 Factorising, 6.7 Factorising, 6.8 Application of Factor Theorem, 6.9 H.C.F. of Polynomials, 6.10 L.C.M. of Polynomials.

7. LINEAR EQUATIONS IN ONE VARIABLE **94—102**

7.1 Linear Equations, 7.2 Applications,

8. LOGARITHMS **103—127**

8.1 Indices, 8.2 Rational Indices, 8.3 Meaning of Logarithms, 8.4 Laws of Logarithms, 8 5 Common Logarithms, 8 6 Characteristic and Mantissa, 8.7 Operations with Logarithms, 8.8 Use of Logarithm Tables, 8.9 Antilogarithms, 8.10 Use of Logarithms, 8.11 Applications.

9. TRIGONOMETRY **128—149**

9.1 Trigonometry, 9.2 Angle in Trigonometry, 9.3 Definitions, 9.4 Trigonometrical Ratios, 9.5 Trigonometrical Ratios of Certain Angles, 9.6 Heights and Distances.

10. STATISTICS 150—171

10.1 Statistics, 10.2 Collection of Data, 10.3 Classification, 10.4 Tabulation, 10.5 Frequency Distribution, 10.6 Statistical Graphs, 10.7 Bar Graphs 10.8 Graphs for Frequency Distributions, 10. Histograms, 10.10 Frequency Polygon, 10.11 Cumulative Frequency Curve.

11. BASIC GEOMETRICAL CONCEPTS 172—187

11.1 Historical Background, 11.2 Importance of the Study of Geometry, 11.3 Nature of Formal Geometry, 11.4 Point, Line and Plane, 11.5 Line-Segment and Ray, 11.6 Parallel and Intersecting Lines, 11.7 Some Definitions, 11.8 Congruence of Line Segments, 11.9 Betweeness, 11.10 Half Plane, 11.11 Angles, 11.12 Kinds of Angles.

12. ANGLE RELATIONS 188—216

12.1 Angle Relations, 12.2 Vertically Opposite Angles, 12.3 Angles made by a Transversal, 12.4 Triangle and its Angles, 12.5 Exterior Angles of a Triangle, 12.6 Quadrilateral, 12.7 Polygons.

13. CONGRUENCE OF TRIANGLES 217—236

13.1 Congruency, 13.2 Congruent Triangles, 13.3 Inequality Relations in a Triangle.

14. PARALLELOGRAMS 237—253

14.1 Quadrilateral, 14.2 Trapezium and Parallelogram, 14.3 Properties of a Parallelogram, 14.4 Derivations of some useful facts about Triangles, 14.5 Intercepts.

15. LOCI AND CONCURRENCY 254—265

15.1 Locus, 15.2 Concurrent Lines.

16. AREA 266—278

16.1 Area, 16.2 Polygonal Regions, 16.3 Area of a Polygonal Region, 16.4 Altitude or Height, 16.5 Between the same Parallels.

17. GEOMETRICAL CONSTRUCTIONS 279—303

17.1 Basic Constructions, 17.2 Elementary Constructions, 17.3 Construction of Triangles, 17.4 Construction of Triangles from other Data, 17.5 Construction of Parallelograms, 17.6 Construction of Rhombuses, 17.7 Construction of Squares, 17.8 Construction of Rectangles, 17.9 Construction of Trapeziums 17.10 Construction of Equivalent Area.

18. LINEAR SYMMETRY 304—317

18.1 Line of Symmetry, 18.2 Symmetry of two points with respect to a line, 18.3 Symmetric Figures.

19. AN INTRODUCTION TO THE COMPUTER 318—340

Mathematics and Computers, What Computers can do ? What Computers can't do ? How is a Computer different from a calculator ? Characteristics of Computers, Problem Solving Mathematics, Algorithm and Computers, Stepwise Refinement of Algorithms, Flow Charts.

TEST PAPERS 341—345

ANSWERS 346—373

1

SETS

1.1. SETS

In our everyday life we come across words that convey the idea of a *collection* or a *set*, for example a pack of cards, a team of players, a bunch of keys, a herd of cattle. The objects that form a set may be different such as '*the children in a class*' or '*players in a team*'. The individual players in '*the set of players in a team*' are the *elements* or *members* of the set.

Sometimes a set may be described completely by listing or naming its elements. We write elements within braces separating them by commas.

The set of odd numbers less than 10 or $\{1, 3, 5, 7, 9\}$.

The set of vowels in the English alphabet or $\{a, e, i, o, u\}$.

Sometimes, it is difficult to list all the elements of a very large set. Such sets are written as follows :

(*i*) The set of counting numbers less than 100. $\{1, 2, 3, 4, 5, 6, 7, 8, ..., 99\}$

(*ii*) The set of letters of the English alphabet. $\{a, b, c, d, e, f, \ldots\ldots, z\}$

Consider the set of all honest people in Delhi.

Can you name the members of this set ?

The property of '*being honest*' cannot be specified exactly. So different persons will have different ideas about it. We say that the *set of honest persons in Delhi is not well-defined.*

The sets used in mathematics are well-defined. We say that **a set is a well-defined collection of objects or things.**

In this book, we will be mainly dealing with sets of mathematical objects like numbers, points, lines, triangles, circles and so on.

We denote sets by *capital letters* A, B, C, P, X, Y, etc. The elements of a set are generally denoted by *small letters* a, b, c, d, x, y, etc.

Let M be the set of all natural numbers less than 25. Then 10 is an element of this set. 16 is also an element of this set. The fact that 16 is an element of the set M is expressed in symbols as $16 \in M$'.

The symble $\in$ is used of the phrases '*is a member*' or '*belongs to*'.

Thus $a \in \{a, e, i, o, u\}$; $7 \in \{2, 3, 5, 7, 9\}$

$k \in A$ is read as '*k is an element of set A*' or '*k belongs to set A*'.

The symbol $\notin$ denotes '*is not a member of*' or '*does not belong to*'.

We see that 30 is not an element of the above set M. We express this fact in symbols as '$30 \notin M$'.

Thus $d \notin \{a, e, i, o, u\}$ $8 \notin \{2, 3, 5, 7, 9\}$

There are two methods of representing sets.

(1) Roster method,

(2) Set builder or rule method.

In roster form all the elements of the set are listed, the elements being separated by commas, and are enclosed within braces.

Let us consider the set of all odd natural numbers less than 10. The elements of this set are 1, 3, 5, 7 and 9. We write this set in roster form as $\{1, 3, 5, 7, 9\}$.

While describing a set in roster form you must remember that every element is listed *only once* and the order in which the elements are listed is *immaterial.*

The above set can also be described as $\{9, 7, 5, 3, 1\}$ or $\{1, 7, 3, 9, 5\}$ or $\{3, 9, 5, 1, 7\}$.

In set builder form the rule is stated according to which elements of the set can be selected. This form is used when all the elements of a set possess a single common property which is not possessed by any element outside the set.

Let us consider the set $\{2, 3, 5, 7, 11, 13, 17, 19\}$. The elements of the set are prime numbers. The last prime number is less than 20. We write this set in set builder form as

$\{x : x$ is a prime number less than $20\}$. It is read as '*the set of all x such that x is a prime number less than 20*'.

Study the following examples :

Roster Method	**Set Builder Method**
$P=\{a, e, i, o, u\}$	$P=\{x : x$ is a vowel$\}$
$T=\{1, 3, 5, 7, 9\}$	$T=\{x : x$ is an odd natural number less than $10\}$
$D=\{7, 14, 21, 28, 35, ..\}$	$D=\{x : x$ is a multiple of $7\}$

Symbol : is used for '*such that*'. The vertical line ' | ' can also be used for '*such that*'.

Example 1. Represent the following set in the roaster form :

$$A=\{x : x \text{ is an integer between 5 and 11}\}.$$

Solution. x is an integer between 5 and 11. The integers between 5 and 11 are 6, 7, 8, 9, 10.

$$\therefore \quad A=\{6, 7, 8, 9, 10\}$$

Example 2. Represent the following set in the set builder form :

$$B=\{3, 4, 5, 6, 7, 8, 9, 10, 11\}.$$

Solution. Let x be the variable to represent elements. Then x is a natural number between 2 and 12.

$$\therefore \quad B=\{x : x \text{ is a natural number between 2 and 12}\}.$$

This can also be written as

$$B=\{x : x \text{ is a natural number and } 2<x<12\}.$$

Example 3. Let M be the set of even numbers between 1 and 11. Which of the following are elements of set M ?

(*a*) 5 (*b*) 8 (*c*) 12

Solution. Even numbers between 1 and 11 are 2, 4, 6, 8, 10.

Then $M=\{2, 4, 6, 8, 10\}$.

(*a*) 5 is not the element of the set M. $\therefore \quad 5 \notin M$

(*b*) 8 is an element of the set M. $\therefore \quad 8 \in M$

(*c*) 12 is an even number, but it is not the element of the set M. $\therefore \quad 12 \notin M$

EXERCISE—1(a)

(Section—A)

1. Which of the following are well-defined sets ?
(*a*) Set of big cities.
(*b*) Set of even prime numbers.
(*c*) Set of factors of 12.
(*d*) Set of months in a year.
(*e*) Set of good books in your school library.

2. Represent the following sets by listing their elements :
(*a*) The set of all one-digit integers.
(*b*) The set of factors of 24.
(*c*) The set of prime numbers upto 10.
(*d*) The set of multiples of 5.

3. List the members of the following sets :
(*a*) $P=\{x : x \text{ is an even prime number}\}$.
(*b*) $S=\{x : x \text{ is a prime factor of } 36\}$.
(*c*) $T=\{x \mid x \in N,\ 5 \leqslant x \leqslant 10\}$.
(*d*) $M=\{x \mid x \text{ is a square of a one-digit natural number}\}$.
(*e*) $A=\{x \mid x \in I,\ x \leqslant 5\}$.

4. State whether the following statements are true or false :
(*a*) $7 \notin \{x \mid x \text{ is an even integer}\}$.
(*b*) $22 \in \{x \mid x \text{ is a perfect square}\}$.

5. State which of the following is false as appropriate :
(*i*) $12 \in \{x : x \text{ is a multiple of } 3\}$.
(*ii*) $2 \in \{x : 2x+2=5\}$.

(Section—B)

6. List the elements of the following sets :
(*a*) $\{x \notin Z : 3x-2=3\}$
(*b*) $\{x : 1 \leqslant x \leqslant 5 \text{ and } x \text{ is a natural number}\}$.

7. Write the set of all natural numbers between 7 and 21 in the set-builder notation.

8. Represent the following sets in the set builder form :
(*a*) $X=\{2, 3, 5, 7, 11, 13, \}$.
(*b*) $E=\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$.
(*c*) $J=\{-3, -2, -1, 0, 1, 2, 3\}$.
(*d*) $K=\{13, 14, 15, 16, 17, 18, 19, 20\}$.

9. Write the set $\{2, 4, 6, 8, \ldots\}$ in the set builder form.

(Section—C)

10. Denote the set $\{x \mid x \text{ is an integer and } x^2=x\}$ by listing its elements.

11. Rewrite the following set by listing all its elements :
$\{x \mid x \in N,\ x \text{ is odd and } x \text{ is a factor of } 90\}$.

1.2. CARDINAL NUMBER

The number of elements in a set is called the cardinal number of the set.

The cardinal number of a set A is *denoted* by $n(A)$.

$n(P)$ means '*the number of elements in the set P*'.

The cardinal number tells the '*how many*' of a set. But ordinal number tells the '*which one*' in a set.

The cardinal numbers are 0, 1, 2, 3, 4, 5, ... as distinguished from ordinal numbers, first, second, third, fourth, fifth, ...

1.3. FINITE AND INFINITE SETS

Consider the set of boys in your class. Each boy can be assigned a counting number taken in order as 1, 2, 3, 4, 5, ..., 27. The number of elements (boys) in the set (class) could be counted and the process of counting is bound to come to an end, sooner or later. Such sets are said to be **finite.**

Every finite set has a cardinal number.

Again consider the set of odd numbers.

$$\begin{array}{lcccccccc} \text{Set of odd numbers} & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15, \ldots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{Set of counting numbers} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

Each odd number can be assigned a counting number taken in order as 1, 2, 3, 4, 5, . .

Does the process of counting come to an end ?

The elements of the set of odd numbers cannot be counted or the process of counting does not come to an end. Such sets are to be **infinite.**

A set is finite if we can find its cardinal number.

A set is infinite if we cannot find its cardinal number.

In the roster form, we write all the elements of the set within braces. It is not possible to write all the elements of an infinite set within braces. So we represent an infinite set in the roster form by writing a few elements which clearly indicate the structure of the set, followed (or preceded) by *three dots.*

We write the set of natural numbers as $\{1, 2, 3, 4, 5, 6, \ldots\}$.

We write the set of integers as

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

1.4. EQUAL SETS

Consider the following sets :

$A=\{x, y, z\}$; $B=\{1, 2, 3\}$

$C=\{x, a, z\}$; $D=\{3, 1, 2\}$

The elements of set A are *not* the same as the elements in set B. We say that sets A and B are different.

Similarly sets A and C are different.

Set B and set D have exactly the *same* elements. We say that sets B and D are **equal.**

It does not make any difference if the elements of the sets are listed in the same order or not.

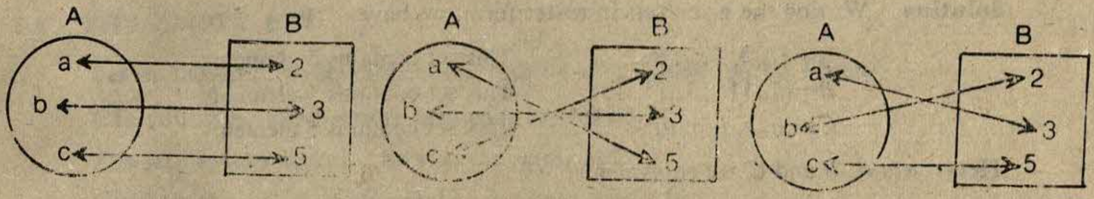
Two sets A and B are said to be equal if every member of A is in B and conversely, every member of B is in A.

Symbolically, we write as $A=B$. If they are not equal, we write as $A \neq B$.

If two sets A and B are equal, it follows that $n(A)=n(B)$, but the converse is not true. If $n(A)=n(B)$, then it is *not* necessary that $A=B$.

1.5. EQUIVALENT SETS

The elements of two sets A and B can be paired in different ways such that each element of one set is associated with a single element of the other with none left over. We say that set A and B have *ane-to-one correspondence.*



Two sets are said to be equivalent sets if there exists a one-to-one correspondence between the elements of the two sets.

Symbolically we write this as $A \leftarrow\rightarrow B$.

Remember that *equal sets are necessarily equivalent but equivalent sets are not necessarily equal.*

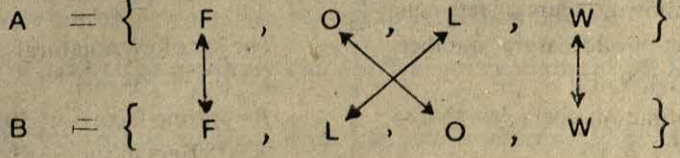
Example 1. Are the following sets equal ?

$A=\{x : x$ is a letter in the word 'FOLLOW'$\}$

$B=\{x : x$ is a letter in the word 'FLOW'$\}$.

Solution. In set A elements 'L' and 'O' occur twice. So each is to be considered only once in listing.

Writing given sets in the tabular form, we have.



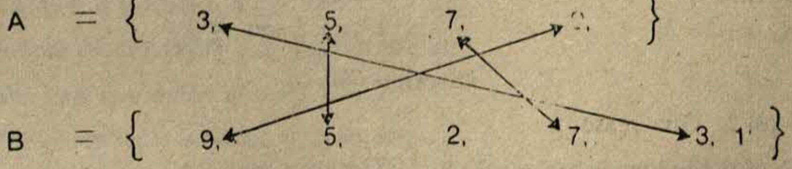
We find that every element of A is also an element of B and every element of B is also an element of A.

Hence sets A and B are equal *i.e.*, $A=B$.

Example 2. State whether the sets

$A=\{3, 5, 7, 9\}$ and $B=\{9, 5, 2, 7, 3, 1\}$ are equal.

Solution. We find that every element of set A is also an element of set B, but every element of set B is not an element of set A.



Hence sets A and B are not equal.

Example 3. Which of the following sets are equivalent ?

$A=\{x : x$ is an odd natural number less than $10\}$.

$B=\{x : x$ is a prime number and $5<x<23\}$.

$C=\{x : x$ is a vowel in English alphabet$\}$.

Solution Writing the given sets in roster form, we have

$A=\{1, 3, 5, 7, 9\}$ This set contains 5 elements.
$B=\{7, 11, 13, 17, 19\}$ This set contains 5 elements.
$C=\{a, e, i, o, u\}$ This set contains 5 elements.

Hence sets A, B and C are equivalent ?

EXERCISE—1 (b)

(Section—A)

1. Which of the following sets are finite and which are infinite :
   - (*a*) The set of prime numbers less than 1000.
   - (*b*) The set of even natural numbers.
   - (*c*) The set of numbers divisible by 1.
   - (*d*) The set of natural numbers between 13 and 31.
   - (*e*) The set of rational numbers between 1 and 2.
   - (*f*) The set of points on a line segment.
   - (*g*) The set of points on a line.
   - (*h*) The set of all hairs on your head.
2. Are the following pairs of sets equal ?
   - (*a*) The set of odd natural numbers ; The set of even natural numbers.
   - (*b*) $X=\{1, 4, 9, 16\}$; $Y=\{1^2, 2^2, 3^2, 4^2\}$
   - (*c*) $A=$\{prime numbers less than 6\} ; $B=$\{prime factors of 30\}
   - (*d*) $J=\{f, l, o, w\}$; $K=$\{letters in the word 'follow'\}
3. Which of the following pairs of sets are equivalent ?
   - (*a*) $S=\{a, e, i, o, u\}$; $T=\{1, 3, 5, 7, 9\}$
   - (*b*) $J=\{1, 2, 3, 4, 5\}$; $K=\{x \mid x \in N, 1<x<6\}$
   - (*c*) $E=$\{the months of the year with 31 days\} : $F=$\{the days of the week\}
4. State whether the following pairs of sets represent equal sets or not.
   - (*a*) $\{a, d, k, m\}$; $\{k, a, m, d\}$
   - (*b*) $\{1, 2, 3, 4\}$; $\{2, 4, 6, 8\}$
   - (*c*) $\{x : x+2=3\}$; $\{x : 1<x<2 \text{ and } x \in N\}$

(Section—B)

5. Is $\phi=\{\phi\}$? Give reasons.
6. Which of the following are equal sets ? Give reasons.

   $\{0\}$, $\{\phi\}$, ϕ, 0
7. Given that $A=\{x \mid x \text{ is a square}\}$ and $B=\{x \mid x \text{ is a rectangle}\}$
   State which of the following is true :

   $A=B$ or $A \neq B$.
8. Which of the following sets equivalent ?

   ϕ, $\{0\}$, $\{\phi\}$.

1.6. THE EMPTY SET

Let us consider the set of all those Indians who landed on the Moon.

Can you name any member belonging to this set ?

The set has *no* element. So it is the *empty set.*

Consider the set of whole numbers between 9 and 10.

There is *no* whole number between 9 and 10. So it is the *empty set.*

The set which contains no element at all is called an empty set or null set.

The symbol ϕ *(phai)* is used to denote a null set or an empty set.

Remember that the cardinal number of the empty set is zero.

Note that $\{0\}$ is *not* the empty set because the number 0 is a member of this set.

1.7. SUBSETS

Consider the following sets :

$$D=\{a, e, i, o, u\} \qquad E=\{o, e, a\}$$

We note that every element of set E is also an element of D. We say, set E is a *subset* of D.

The set of students in your class is a subset of 'the set of all the students in your school' because every student in your class belongs to the school also.

The set A is said to be a subset of a set B if every element of A is also an element of B.

Symbolically, we write $A \subseteq B$ which is read as '*A is a subset of B*' or '*A is included in B*'.

Always remember that the null set is a subset of any set.

Consider all the possible subsets of $\{a, b, c\}$.

No-element subset	ϕ
One-element subsets	$\{a\}, \{b\}, \{c\}$
Two-element subsets	$\{a, b\}, \{b, c\}, \{c, a\}$
Three-element subset	$\{a, b, c\}$

The number of subsets is 8 *i.e.*, 2^3.

Let us note the following points about subsets.

(*i*) *Every set is a subset of itself.*

(*ii*) *The empty set is a subset of every set*

(*iii*) *A set with n elements has 2^n subsets.*

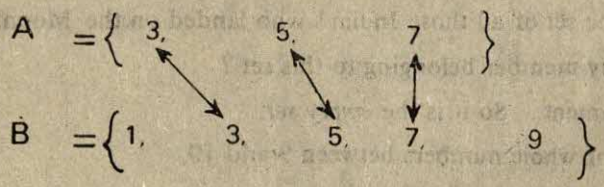
When a subset of a set is not equal to the set, the subset is called a *proper subset.*

Thus $\{a\}, \{a, b\}, \{b, c\}$ are proper subsets of the set $\{a, b, c\}$ but the subset $\{a, b, c\}$ is *not* a proper subset.

We use the symbol $\subset$ to write a proper subset.

Example 1. Is $A=\{3, 5, 7\}$ a subset of $B=\{1, 3, 5, 7, 9\}$?

Solution.



We find that every element of set A is also an element of set B. Therefore, set A is subset of set B.

$\therefore \qquad A \subset B.$

Example 2. Examine whether

$A = \{x : x \text{ is a positive integer divisible by } 3\}$

is a subset of $\quad B = \{x : x \text{ is a positive integer divisible by } 5\}$.

Solution. Writing the given sets in tabular form, we have

$A = \{3, 6, 9, 12, 15, 18, \ldots\}$

$B = \{5, 10, 15, 20, 25, 30, \ldots\}$

Here 3 is an element of A, but it is not an element of B.

Also 6 is an element of A, but it is not an element of B.

Thus, we find that every element of set A is not an element of set B. Therefore A is not a subset of B.

$\therefore \qquad A \not\subset B$

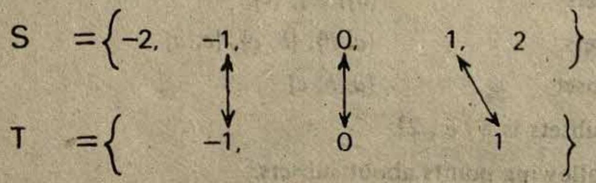
Example 3. Let $\quad S = \{-2, -1, 0, 1, 2\}$, and

$T = \{x : x \text{ is an integer and } -2 < x < 2\}$

Is S a subset of T?

Is T a subset of S?

Solution. Rewriting the given sets, we have



We find that every element of S is not an element of T. Therefore S is not a subset of T i.e., $S \not\subset T$.

Here every element of T is also an element of S. Therefore T is a subset of S i.e., $T \subset S$.

Let us consider the following sets :

$A = \{1, 3, 5, 7, 9\}$

$B = \{x : x \text{ is an odd natural number } < 10\}$

Is A a subset of B? $\qquad$ Is B a subset of A?

Here A is subset of B and B is a subset of A. We also find that A and B are equal sets. It is always true.

If A is subset of B and B is a subset of A, then A=B. Conversely, if A is equal to B, then A is a subset of B and B is a subset of A.

In symbols, $A=B$, if and only if $A \subset B$ and $B \subset A$.

1.8. DISJOINT SETS

Consider the following sets :

$A=\{a, e, i, o, u\}$ $\qquad$ $B=\{d, p, f, t, x\}$

Sets A and B have *no* elements in common. We say that A and B are **disjoint sets.**

Two sets are said to be disjoint, if they have no element in common.

$\{1, 3, 5, 7, 9, ...\}$ and $\{2, 4, 6, 8, 10, ...\}$ are disjoint sets.

EXERCISE—1 (c)

(Section—A)

1. Which of the following are examples of the empty set ?
   - (*a*) The set of months which start with the letter *J*.
   - (*b*) The set of prime numbers divisible by 2.
   - (*c*) The set of points of intersection of two parallel lines.
   - (*d*) The set of even numbers divisible by 3.
   - (*e*) The set of quadrilaterals with five sides.
   - (*f*) The set of counting numbers that satisfy the equation $x^2+1=0$.
2. Let $A=\{1, 2, 3, 4, 5, 6, 7, ...\}$
   - (*a*) Is the set of odd numbers a subset of A ?
   - (*b*) Is the set of even numbers a subset of A ?
3. List the all subsets of $\{1, 2, 3\}$.
4. List all the subsets of $\{0, 1\}$.
5. State whether each of the following statements is true or false :
   - (*a*) $\{x : x \text{ is an odd number}\} \subset \{x : x \text{ is an integer}\}$
   - (*b*) $\{x : x \text{ is a multiple of } 3\} \not\subset \{x : x \text{ is a multiple of } 6\}$
   - (*c*) $\{x : x \text{ is a positive integer divisible by } 3\} \subset \{x : x \text{ is a positive integer divisible by } 4\}$
6. What is the total number of subsets of the set $\{1, 3, 5, 6\}$?
7. Given that A is a subset of B and B is a subset of A.
   Are sets A and B equal ?
   Illustrate your answer with examples.
8. Which of these sets are disjoint ?
   - (*a*) $\{a, e, i, o, u\}$ and $\{a, b, c, d, e\}$
   - (*b*) The set of even counting numbers ; The set of odd counting numbers.
   - (*c*) $X=\{x : x \text{ is a perfect square}\}$; $Y=\{x : x \text{ is a perfect cube}\}$
   - (*d*) $A=\{x : x \text{ is a positive integer}\}$; $B=\{x : x \text{ is a negative integer}\}$
9. In answer to each of the following statements, write true or false as appropriate :
   - (*a*) If $A \subset B$ and $C \subset B$, then $A \subset C$.
   - (*b*) The total number of subsets of a finite set which contains n elements is 2^n.

(Section—B)

10. Let $B=\{2, 4, 6, 8\}$, $D=\{3, 4, 5\}$, $E=\{3, 5\}$

Determine a set F such that

(*a*) $F \subset D$, $F \not\subset B$ (*b*) $F \subset E$, $F \not\subset B$

(*c*) $F \subset D$, $F \subset E$ (*d*) F and D are disjoint.

1.9. UNIVERSAL SET

Consider the following sets :

A=The set of boys studying in your school.

B=The set of girls studying in your school.

C=The set of students of your school using school-bus.

D=The set of prefects in your school.

Who are the members of these sets ?

The students of your school form a bigger set out of which other sets are formed.

The members of each of these sets are taken from '*the set of all students in your school*'.

We call '*the set of all students in your school*' the **universal set** for sets A, B, C and D. Sets A, B, C and D are the *subsets* of this universal set.

A set which contains all subsets is often called a universal set or in other words, **the set from which we select elements to form any given set is called the universal set.**

We denote the universal set by E or U, or ξ.

1.10. COMPLEMENT OF A SET

Consider the following sets :

$U=\{a, b, c, d, e, f, g\}$ $A=\{b, d, f\}$

Set A is a proper subset of the universal set U.

There are elements of U which are not elements of set A. These remaining elements form a new set $\{a, c, e, g\}$. We call this set the **complement of** set A.

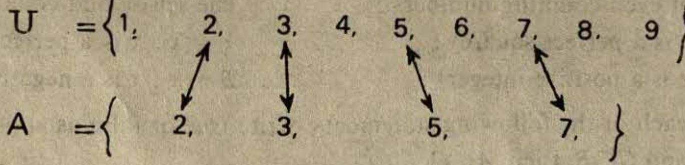
Let U be the universal set and let A be the given set. The complement of A is the set of all elements of U which are not elements of A.

Given a universal set E, $A'=\{x \mid x \in E \text{ and } x \notin A\}$

We denote the complement of A by A'. This is read '*A dash*' or '*A prime*'.

Example 1. Let $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and
$A=\{2, 3, 5, 7\}$, find A'.

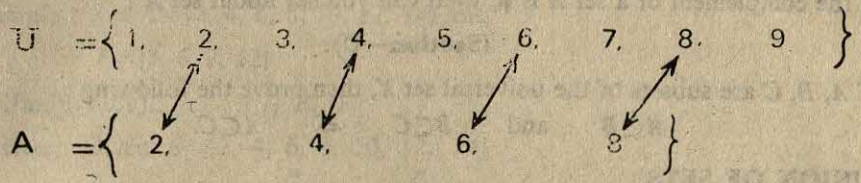
Solution.



$\therefore$ A'=The set of elements in U which are not in A.
$=\{1, 4, 6, 8, 9\}$.

Example 2. If $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find the complement of A when $A=\{2, 4, 6, 8\}$. Also find (A').

Solution.



$\therefore$ A' = The set of elements in U which are not in $A\}$
$=\{1, 3, 5, 7, 9\}$.

Now $(A')'$ = The set of elements in U which are not in A'.
$=\{x : x \in U \text{ and } x \notin A'\}$
$=\{2, 4, 6, 8\}$
$=A$.

What do you note ?

The complement of the complement of a set is the set itself.

EXERCISE—1 (d)

(Section—A)

1. Give a suitable Universal set for the following sets :
   The sets of right triangles
   The set of equilateral triangles, and
   The set of isosceles triangles.
2. Write a universal set for each of the following sets :
   (*a*) $\{a, e, i, o, u\}$ (*b*) $\{4, 9, 16, 25, 36, \ldots\ldots\}$
   (*c*) $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
3. Given $U=\{a, b, c, d, e, f\}$
   $A=\{b, d, e\}$ $B=\{a, f\}$ $C=\phi$
   Find A', B' and C'
4. Given $U=\{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A=\{1, 2, 3, 4\}$.
   What is A', the complement of set A ?
5. Let U, the set of natural numbers, be the universal set.
   Write the complement of each of the following sets :
   (*a*) $\{x : x$ is a prime number$\}$
   (*b*) $\{x : x$ is a perfect square$\}$
   (*c*) $\{x : x$ is a number divisible by 3 and 5$\}$.

(Section—B)

6. $U=\{1, 2, 3, 6, 7, 12, 17, 21, 35, 52, 56\}$ and P, Q and R are the subsets of U such that
   P={odd numbers}
   Q={numbers divisible by 7}
   R={prime numbers}

List the elements of the set :

(*a*) R (*b*) Q' (*c*) P'

7. If the complement of a set A is ϕ, what can you say about set A ?

(Section—C)

8. If A, B, C are subsets of the universal set X, then prove the following :

$$A \subset B \quad \text{and} \quad B \subset C \quad \Rightarrow \quad A \subset C.$$

1.11. UNION OF SETS

Consider the following sets :

$A=\{a, b, c, d\}$ $B=\{e, f, g\}$ $C=\{a, b, c, d, e, f, g\}$

Is every element of set A also an element of set C ?

Is every element of set B also an element of set C ?

We see that set C contains all the elements of sets A and B both. We say that C is the **union** of A and B.

Consider the following sets :

$D=\{a, b, c, d, e\}$ $E=\{c, d, e, f, g\}$ $F=\{a, b, c, d, e, f, g\}$

We see that every element of set F is an element of either D or E or both. We say that F is the **union** of D and E.

The union of two sets A and B is a set of all elements which belong to A or to B or to both A and B.

The union of two sets A and B is denoted symbolically as $A \cup B$. It is read as '*A union B*'.

We can represent $A \cup B$ in the set builder form as follows :

$$A \cup B=\{x : x \in A \vee x \in B\}$$

Note that $A \subset A \cup B$ and $B \subset A \cup B$

1.12. INTERSECTION OF SETS

Consider the following sets :

$A=\{a, b, c, d, e\}$ $B=\{c, d, e, f, g\}$ $C=\{c, d, e\}$

Is every element of set C also an element of set A ?

Is every element of set C also an element of set B ?

We see that C contains all the elements which are common to both A and B. We say that C is the **intersection** of A and B.

The intersection of two sets A and B is a set of all elements which belong to A as well as to B.

The intersection of two sets A and B is denoted symbolically as $A \cap B$. It is read as '*A intersection B*'.

We can represent $A \cap B$ in the set builder form as follows :

$$A \cap B=\{x : x \in A \wedge x \in B\}$$

Consider the following sets :

$C=\{a, e, i, o, u\}$ $D=\{b, d, f, k\}$

We see that there is *no* element common to both C and D. The intersection of C and D contains *no* element. Thus $C \cap D = \phi$. Hence sets C and D are ***disjoint sets.***

Note that $A \cap B \subset A$ and $A \cap B \subset B$.

Example : Let $E=\{2, 4, 6, 8, 10, 12, 14\}$ and

$F=\{3, 6, 9, 12\}$

Find (*i*) $E \cup F$ (*ii*) $E \cap F$

Solution : Given $E=\{2, 4, 6, 8, 10, 12, 14\}$

↗ ↗

↙ ↙

$F=\{3, 6, 9, 12\}$

(*i*) $E \cup F=\{2, 4, 6, 8, 10, 12, 14, 3, 9\}$

(*ii*) $E \cap F=\{6, 12\}$

EXERCISE—1 (e)

(Section—A)

1. Let $A=\{a, b, c, d, e\}$ and $B=\{b, d, f\}$.
   Find $A \cup B$ and $A \cap B$.
2. Let $C=\{p, q, r, s\}$ and $D=\{x, y, z\}$.
   Find $C \cup D$ and $C \cap D$.
3. If $A=\{2, 4, 6, 8, 10, 12, 14\}$, and
   $B=\{3, 6, 9, 12\}$, find
   (*a*) $A \cup B$ (*b*) $A \cap B$
4. If $E=\{a, i, j, k, l, m, n\}$, and
   $F=\{a, c, e, k, m, n\}$, find
   (*a*) $E \cup F$ (*b*) $E \cap F$
5. If $A=\{x \mid x$ is a multiple of 5 and $x \leqslant 20\}$ and
   $B=\{1, 3, 7, 10, 12, 15, 18, 25\}$, find
   (*i*) $A \cup B$ (*ii*) $A \cap B$.
6. Find the union of the following sets :
   (*a*) $\{x \mid x$ is and odd natural number$\} \cup \{x \mid x$ is an even natural number$\}$
   (*b*) $\{x \mid x$ is a rational number$\} \cup \{x \mid x$ is an irrational number$\}$
   (*c*) $\{x \mid x$ is a multiple of 3$\} \cup \{x \mid x$ is a multiple of 5$\}$
   (*d*) $\{x \mid x$ is a negative integer$\} \cup \{x \mid x$ is a non-negative integer
7. Find the intersection of the following sets :
   (*a*) $\{x \mid x$ is a prime number$\} \cap \{x \mid x$ is an even number$\}$
   (*b*) $\{x \mid x$ is an positive odd integer$\} \cap \{x \mid x$ is a positive even integer$\}$
   (*c*) $\{x \mid x$ is a rational number$\} \cap \{x \mid x$ is an integer$\}$
   (*d*) $\{x \mid x$ is a multiple of 6$\} \cap \{x \mid x$ is a multiple of 3$\}$
8. List the elements of the following set :
   $\{x : x \leqslant -5, x \in z\} \cap \{x : x \geqslant -5, x \in z\}$

(Section—B)

9. If $A=\{3, 8\}$, $B=\{4, 8, 12\}$ and $E=\{2, 3, 4, 6, 8, 10, 12\}$,
find (*a*) $A \cup B$ (*b*) $A' \cap B$ (*c*) $A \cap B'$ (*d*) $A \cap B$

10. $U=\{x : x \leqslant 25, x \in N\}$, $A=\{x : x \leqslant 15\}$ and
$B=\{x : 10 < x \leqslant 25\}$: list the elements of the following sets :
(*i*) A' (*ii*) B' (*iii*) $A' \cup B'$ (*iv*) $A' \cap B'$

11. Prove that A and B are subsets of $A \cup B$.

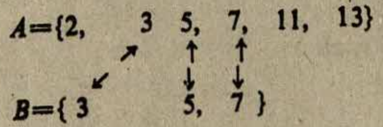
12. Prove that $A \cap B$ is subset of A.

(Section—C)

13. If $A=\{3, 5, 7, 9, 11\}$, $B=\{7, 9, 11, 13\}$, and
$C=\{11, 13, 15\}$, find
(*a*) $A \cap (B \cup C)$ (*b*) $(A \cap B) \cap C$

Let us learn more about operations of union and intersection of sets.

Let $A=\{2, 3, 5, 7, 11, 13\}$ and $B=\{3, 5, 7\}$



we find that $B \subset A$

$$A \cup B=\{2, 3, 5, 7, 11, 13\} \cup \{3, 5, 7\}$$
$$=\{2, 3, 5, 7, 11, 13\}$$

$\therefore$ $A \cup B=A$, when $B \subset A$.

What do you note ?

$$A \cap B=\{2, 3, 5, 7, 11, 13\} \cap \{3, 5, 7\}$$
$$=\{3, 5, 7\}$$

$\therefore$ $A \cap B=B$, when $B \subset A$.

What do you note ?

1.13. PROPERTIES OF OPERATIONS WITH SETS

The operations of union and intersection on sets have some similarity with the operations of addition and multiplication in real numbers.

Let us consider the following three sets :
$A=\{a, b, c, d\}$, $B=\{c, d, e, f, g\}$, $C=\{g, h, a\}$

(*i*) We find the union of two sets at a time. So union is a *binary operation.*

(*ii*) $A \cup B=\{a, b, c, d\} \cup \{c, d, e, f, g\} =\{a, b, c, d, e, f, g\}$
$B \cup A=\{c, d, e, f, g\} \cup \{a, b, c, d\} =\{c, d, e, f, g, a, b\}$

We see that the two sets so formed consist of the same elements.

$\therefore$ $A \cup B=B \cup A$

The order in which we form the union of two sets *does not* affect the union set. This is called **commutative property for union of sets.**

(*iii*) Union of three or more sets :

$(A\cup B)\cup C=\{a, b, c, d\}\cup\{c, d, e, f, g\}\cup\{g, h, a\}$
$=\{a, b, c, d, e, f, g\}\cup\{g, h, a\}$
$=\{a, b, c, d, e, f, g, h\}$

$A\cup(B\cup C)=\{a, b, c, d\}\cup\{c, d, e, f, g\}\cup\{g, h, a\}$
$=\{a, b, c, d\}\cup\{c, d, e, f, g, h, a\}$
$=\{a, b, c, d, e, f, g, h\}$

We observe that the two sets so formed consist of the same elements.

$\therefore \quad (A\cup B)\cup C=A\cup(B\cup C)$

The manner in which we group two of the three sets in forming their union *does not* affect their union. This is called **associative property for union of sets**.

(*iv*) $A\cup\phi=\{a, b, c, d\}\cup\{\ \}$
$=\{a, b, c, d\} \quad =A$

Thus $A\cup\phi=A$

ϕ is called the **identity element of union**.

(*v*) $A\cup A=\{a, b, c, d\}\cup\{a, b, c, d\}$
$=\{a, b, c, d\} \quad =A$

Thus $A\cup A=A$

This is known as **Idempotent law for union.**

(1) We find the intersection of two sets at a time. So intersection is a **binary operation**.

(2) $A\cap B=\{a, b, c, d\}\cap\{c, d, e, f, g\}$
$=\{c, d\}$
$B\cap A=\{c, d, e, f, g\}\cap\{a, b, c, d\}$
$=\{c, d\}$

Since these sets consist of the same elements, they are equal.

$\therefore \quad A\cap B=B\cap A$

The order in which we form the intersection of two sets *does not* affect the intersection set. This is called **commutative property for intersection of sets.**

(3) $(A\cap B)\cap C=\{a, b, c, d\}\cap\{c, d, e, f, g\}\cap\{a, g, h\}$
$=\{c, d\}\cap\{a, g, h\}$
$=\{\ \} \quad =\phi$

$A\cap(B\cap C)=\{a, b, c, d\}\cap\{c, d, e, f, g\}\cap\{a, g, h\}$
$=\{a, b, c, d\}\cap\{g\}$
$=\{\ \} \quad =\phi$

Since these sets consist of the same elements, they are equal.

$\therefore \quad (A\cap B)\cap C=A\cap(B\cap C)$

The manner in which we group the sets in forming their intersection, *does not* affect their intersection. This is called the **associative property for intersection of sets.**

(4) $A\cap U=\{a, b, c, d\}\cap\{a, b, c, d, e, \ldots, z\}$
$=\{a, b, c, d\} \qquad =A$

Thus $A\cap U=A$

U is called the **identity element of intersection.**

(5) $A\cap A=\{a, b, c, d\}\cap\{a, b, c, d\}$
$=\{a, b, c, d\}=A$

Thus $A\cap A=A$

This is known as **Idempotent law for intersection.**

Let $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$A=\{2, 4, 6, 8\}$ and $B=\{2, 3, 5, 7\}$

Then $A'=\{1, 3, 5, 7, 9\}$ and $B'=\{1, 4, 6, 8, 9\}$

(*i*) $A\cup B=\{2, 4, 6, 8\}\cup\{2, 3, 5, 7\}$
$=\{2, 3, 4, 5, 6, 7, 8\}$

Then $(A\cup B)'=\{1, 9\}$
and $A'\cap B'=\{1, 3, 5, 7, 9\}\cap\{1, 4, 6, 8, 9\}$
or $A'\cap B'=\{1, 9\}$

Since three sets consist of the same elements, they are equal.

$\therefore \quad (A\cup B)'=A'\cap B'$

What do you note ? Take other sets and verify the result.

(*ii*) $A\cap B=\{2, 4, 6, 8\}\cap\{2, 3, 5, 7\}$
$=\{2\}$

Then $(A\cap B)'=\{1, 3, 4, 5, 6, 7, 8, 9\}$
and $A'\cup B'=\{1, 3, 5, 7, 9\}\cup\{1, 4, 6, 8, 9\}$
or $A'\cup B'=\{1, 3, 4, 5, 6, 7, 8, 9\}$

Since these sets consist of the same elements, so they are equal.

$\therefore \quad (A\cap B)'=A'\cup B'$

What do you note ?

Take other sets and verify the result.

Thus $(A\cup B)'=A'\cap B'$; $(A\cap B)'=A'\cup B'$

These are known as **De Morgan's Laws.**

EXERCISE—1 (f)

(Section—A)

1. If $B\subset A$, prove that $A\cup B=A$.
2. If $B\subset A$, prove that $A\cap B=B$.
3. If $U=\{x : 1<x<20, x\in N\}$,
   $P=\{p : p=2n, n\in N\}$
   $Q=\{q : q=3n, n\in N\}$
   $R=\{r : r=9n, n\in N\}$
   list the following sets :
   (*a*) $P\cap Q$ (*b*) $P\cap Q\cap R$
   (*c*) $P'\cap Q\cap R'$ (*d*) $P\cap Q\cap R'$

4. If $A=\{1, 2, 3, 5\}$, $B=\{2, 4, 5, 6\}$ and $C=\{3, 4, 5, 7\}$,
verify (*a*) $A\cup(B\cup C)=(A\cup B)\cup C$ (*b*) $(A\cap B)\cap C=A\cap(B\cap C)$

5. If $A=\{1, 2, 3\}$, $B=\{2, 3, 4\}$, $C=\{3, 4, 5, 6\}$ and $U=\{1, 2, 3, 4, 5 \ldots\ldots, 9\}$,
show that (*a*) $(A\cup B)'=A'\cap B'$ (*b*) $(A\cap B)'=A'\cup B'$

6. For any two subsets A and B of U, show that

$$(A\cap B)'=A'\cup B'$$

7. For any two subsets X and Y of U, show that

$$(X\cup Y)'=X'\cap Y'$$

(Section—B)

8. If $A=\{x : x \text{ is a multiple of } 3\}$, and $B=\{x : x \text{ is even}\}$
are subsets of the universal set $U=\{x : x \text{ is a natural number} \leqslant 15\}$, then verify that
(*a*) $(A\cap B)'=A'\cup B'$, and (*b*) $(A\cup B)'=A'\cap B'$.

9. Using sets $U=\{x : 0<x\leqslant 25, x\in N\}$,
$A=\{x : x \text{ is a multiple of } 2\}$
$B=\{x : x \text{ is a multiple of } 3\}$ and
$C=\{x : x \text{ is a multiple of } 5\}$,
show that $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

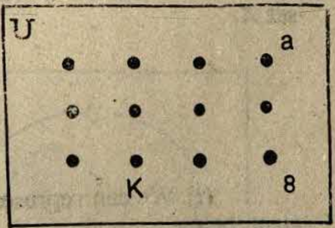
(Section—C)

10. Using sets $U=\{x : x\leqslant 10, x\in N\}$,
$A=\{1, 3, 5, 7, 9\}$, $B=\{2, 4, 6, 8\}$, and
$C=\{2, 3, 4, 5, 6\}$,
verify that $A\cap(B'\cup C)=A\cap(B\cap C)'$

1.14. VENN DIAGRAMS

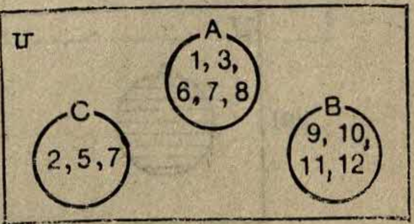
In order to help us in thinking about set relations and operations, we make use of pictorial representations called **Venn-diagrams** named after the English logician **Jonn Venn.**

The universal set ξ is represented by the interior of a rectangle. The elements of the set ξ are represented by points or dots or by writing the elements, as shown on the right.



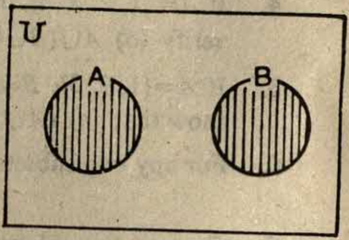
The arbitrary subsets of the universe are represented by the dots or elements enclosed in a circle drawn inside the rectangle.

The subset A has 5 elements, B has elements 9, 10, 11, 12, and C has elements 2, 5, 7.

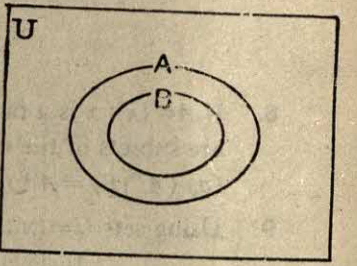


The subsets can also be represented by ellipses.

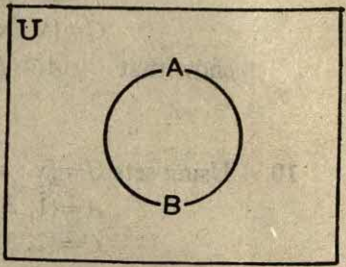
The sets A and B can be represented as in the adjoining figure. These are such that no member of set A is a member of set B and vice-versa. Sets A and B are *disjoint sets.*



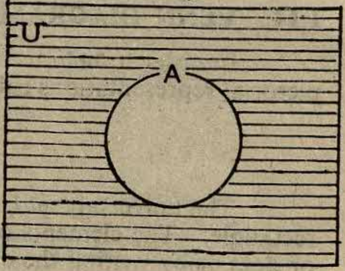
A possible relation may be that set B is inside the set A. It means that all the elements of the set B are the members of the set A also. Thus B is a *proper set* of A. We write '$B \subset A$'.



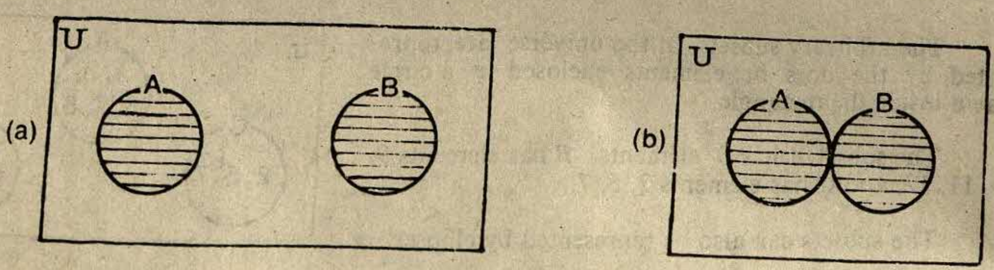
When the set A and the set B are *equal sets,* then the same circle represents A or B. We write $A=B$.



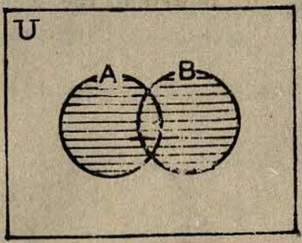
The shaded portion represents the *complement* of set A.



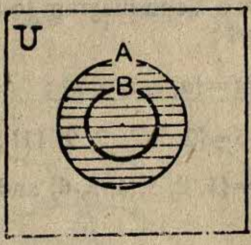
(*i*) We can represent the union of two sets A and B by means of Venn-diagrams as given below :



(c)



(d)

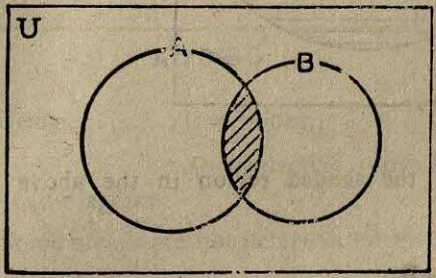


The *shaded* portion represents $A \cup B$.

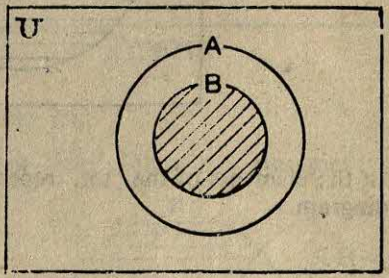
Note that the union of set A and its subset B is the set A itself, as shown in the figure (*d*).

(*ii*) We can represent the intersection of two sets A and B by means of Venn-diagrams as given below :

(a)



(b)



The *shaded* portion represents $A \cap B$ in both figures.

Note that the intersection of set A and its subset B is the subset B itself, as shown in the figure (*b*).

Example. If B and C are two subsets of the universal set U, draw a Venn diagram for the following

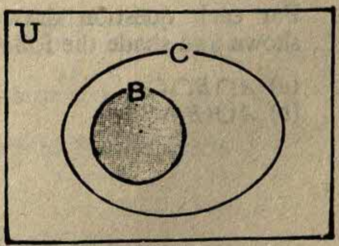
$$B \cap C \text{ when } B \subset C$$

Solution. We represent the universal set U by a rectangle, the subset B by a circle and the subset C by an ellipse.

Since B and C are subsets of U, the ellipse for C and the circle for B are to be draw inside the rectangle.

Since B is a subset of C, the circle for B is to be drawn inside the ellipse for C.

Then $B \cap C$ is the set B which is shown by shaded portion.



EXERCISE 1(g)

(Section—A)

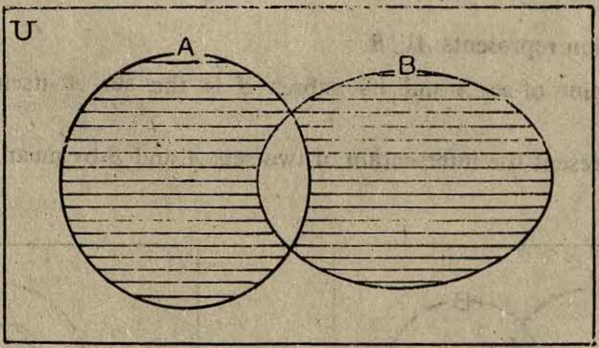
1. Draw a Venn-diagram to illustrate that A is a proper subset of B.
2. Draw a Venn-diagram for two arbitrary sets A and B which are disjoint.

3. Draw a Venn-diagram to show each of the following, to represent the elements of the set :

(*a*) $M=\{a, e, i, o, u\}$; $P=\{e, o, a\}$

(*b*) $S=\{2, 3, 5, 7, 9, 11\}$; $T=\{5, 2, 7\}$

4. If $A=\{1, 2, 3, 4, 5, 6\}$ and $B=\{2, 6, 7, 9, 11\}$,



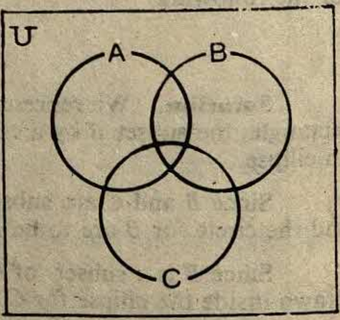
list the elements of the set, represented by the shaded region in the above Venn-diagram.

(Section—B)

5. Draw a Venn-diagram to illustrate each of the following sets and subsets :

$U=$ {all natural numbers} ; $D=$ {even prime numbers}

$E=$ {odd numbers) and $F=$ {even numbers}



6. For each question draw three intersecting circles as shown and shade the following :

(*a*) $A\cap B\cap C$
(*b*) $A\cup(B\cap C)$

7. Draw an appropriate Venn-diagram for each of the following :

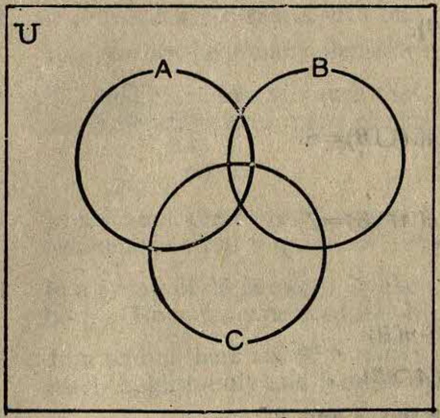
(*a*) $A\subset B$, $B\subset C$, C and D are disjoint sets.

(*b*) A and C are subsets of B, A and C are disjoint sets.

8. Illustrate by a Venn-diagram the relationship between the sets A, B and C, given that $B\subset A$, $C\subset B'$ and $A\cap C=\phi$.

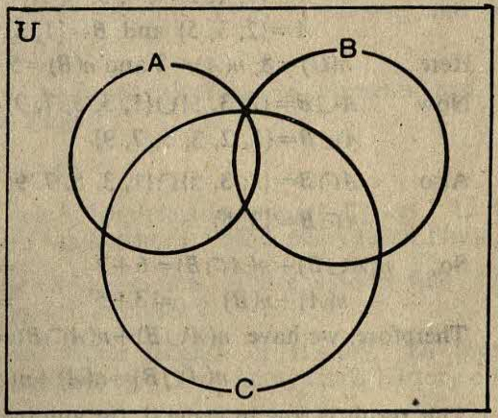
9. In each of the diagrams given below, shade the region which represents the set given underneath the diagrams.

(a)



$A' \cap B \cap C'$

(b)



$A' \cap B' \cap C'$

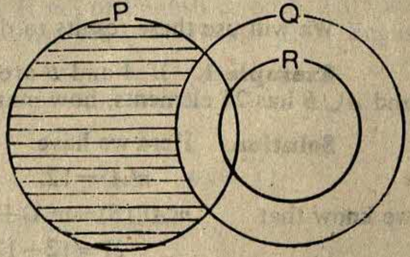
10. Using $U = \{\text{books}\}$, $N = (\text{novels})$, and $D = \{\text{detective novels}\}$, represent the following statement as a Venn-diagram :

'Some novels are not detective novels'.

11. (*i*) Name the set which the shaded region of the given Venn-diagram represents.

(*ii*) In the same diagram shade the region which represents

$(P \cap Q) \cup R$



(Section—C)

12. Given $U = \{x : 1 \leqslant x \leqslant 15, x \in Z\}$

$A = \{p : p \text{ is multiple of } 2\}$, $B = \{q : q \text{ is multiple of } 3\}$,

$C = \{r : r \text{ is a multiple of } 9\}$

Draw a Venn-diagram to show the relationship between these sets.

13. Draw a Venn-diagram to illustrate the relationship among the following sets :

$U = \{\text{natural numbers}\}$, $E = \{\text{even numbers}\}$,

$P = \{\text{prime numbers}\}$, $S = \{\text{square numbers}\}$.

In your diagram shade the region which represents the set of even numbers which are neither prime nor square.

14. Draw a Venn-diagram to show the relationship between the following sets :

$U = \{\text{quadrilaterals}\}$, $A = \{\text{parallelogram}\}$

$B = \{\text{rectangles}\}$, $C = (\text{rhombuses})$.

Shade in your diagram the region that represents the set of squares.

15154

1.15. APPLICATIONS

Let us consider the universal set U and its two subsets A and B where

$$U=\{0, 1, 2, 3, 4, 5, 6, 7\},$$
$$A=\{2, 3, 5\} \text{ and } B=\{1, 3, 5, 7, 9\}.$$

Here $n(U)=8$, $n(A)=3$ and $n(B)=5$

Now $A\cup B=\{2, 3, 5\}\cup\{1, 3, 5, 7, 9\}$

or $A\cup B=\{1, 2, 3, 5, 7, 9\}$ $\quad\therefore\ n(A\cup B)=6$

Also $A\cap B=\{2, 3, 5\}\cap\{1, 3, 5, 7, 9\}$

or $A\cap B=\{3, 5\}$ $\quad\therefore\ n(A\cap B)=2$

So, $n(A\cup B)+n(A\cap B)=6+2 \quad =8$

$n(A)+n(B) \quad =3+5 \quad =8$

Therefore, we have $n(A\cup B)+n(A\cap B)=n(A)+n(B)$

i.e., $$n(A\cup B)=n(A)+n(B)-n(A\cap B).$$

This result is true in general, for any two finite sets A and B.

If A and B are disjoint sets, then $A\cap B=\phi$.

$\therefore$ **$\mathbf{n(A\cup B)=n(A)+n(B)}$**

If A and B are not disjoint sets, then $A\cap B\neq\phi$.

$\therefore$ **$\mathbf{n(A\cup B)=n(A)+n(B)-n(A\cap B)}$**

We will use these results to discuss the applications of sets.

Example 1. If A and B are two sets such that A has 12 elements, B has 17 elements, and $A\cup B$ has 21 elements, how many elements does $A\cap B$ have ?

Solution. Here we have

$$n(A)=12, \quad n(B)=17, \quad n(A\cup B)=21$$

we know that $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$\therefore$ $21=12+17-n(A\cap B)$

or $21=29-n(A\cap B)$

or $n(A\cap B)=29-21$

$=8$

Example 2. Out of 80 pupils who scored first class marks in Mathematics or English, 50 obtained first class marks in Mathematics and 10 in English and Mathematics. How many pupils scored first class marks in English only ? [*C.B.S.E., 1977 (Delhi)*]

Solution. Let the set of pupils scoring first class marks in English be A.

Let the set of pupils scorting first class marks in Mathematics be B.

Then the set of pupils scoring first class marks in English or Mathematics $=A\cup B$.

The set of pupils scoring first class marks in both English and Mathematics $=A\cap B$.

Given $n(A\cup B)=80$, $n(A\cap B)=10$ and $n(B)=50$

We know that $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

$\Rightarrow$ $80=n(A)+50-10$

$\Rightarrow$ $(A)=40$

Thus, number of pupils scoring first class marks in English only $=n(A)-n(A\cap B)$

$=40-10=30.$

EXERCISE—1(b)

(Section—A)

1. If $n(X)=72$, $n(Y)=84$, $n(X\cap Y)=50$, find $n(X\cup Y)$.

2. If A and B are two sets such that $A\cup B$ had 18 elements, A has 8 elements, and B has 15 elements, how many elements does $A\cap B$ have ?

3. If X and Y are two sets such that X has 15 elements, $X\cup Y$ has 16 elements and $X\cap Y$ has 6 elements, how many elements does Y has ?

(Section—B)

4. In a school, there are 20 teachers who teach Mathematics and/or Physics. 12 teach Mathematics and 9 teach both Physics and Mathematics. How many teach Physics ?

5. In a group of 26 persons, 15 like tea while 18 like coffee. There is none who dislikes both. How many persons like both tea as well as coffee ?

6. In a school there are 40 teachers who teach either English or History. Of these 24 teach English only and 8 teach English and History. How many teach History only ?

7. In a colony with population 20,000 ; 10,200 people read 'Indian Express' ; 9,200 read 'Hindustan Times' and 1600 read both. How many read neither ?

8. In a group, members know at least one of the languages, Hindi and Urdu. 100 members know Hindi, 50 members know Urdu and 25 of them know both Hindi and Urdu. How many members are there in the group ?

9. The members of a group of 400 people speak either Hindi or English or both. 270 speak Hindi only and 50 speak both Hindi and English. How many of them speak English only ?

10. Two sets A and B are such that $n(A\cup B)=18$,
$n(A'\cap B)=3$, $n(A\cap B')=5$. Find $n(A\cap B)$.

11. A and B are two sets such that the $A\subset B$, $n(A)=8$, and $n(B)=12$.
If $n(U)=15$, where U is the universal set, find the following :
(*i*) $n(A')$, (*ii*) $n(B')$, (*iii*) $n(A\cap B')$, (*iv*) $(A'\cap B)$

12. Out of 80 students in a science faculty at a university offered Physics, Chemistry and Mathematics as under :
10 offered Mathematics only ; 10 offered Chemistry only ; 6 offered Physics only ; 40 offered Physics and Mathematics ; 20 offered Physics and Chemistry ; 60 offered Physics.
How many students offered Mathematics ?

13. In a school, all pupils play either Hockey or Football or both. 300 pupils play Football, 250 play Hockey and 110 play both games.
Find (*i*) the number of pupils who play Football only ;
(*ii*) the number of pupils who play Hockey only ;
(*iii*) the total number of pupils in the school.

(Section—C)

14. In a certain locality of Delhi there are 1000 families. A survey indicated that 300 subscribe to the 'Hindustan Times' daily newspaper and 250 subscribe to the 'Statesman' daily newspaper and of these two categories 100 subscribe to both. Find the number of families which do not subscribe to any of these newspapers.

REVIEW EXERCISE—I

(Section—A)

1. State whether the following are true or false as appropriate :

(*a*) There are no subsets of the empty set.

(*b*) The cardinal number of the set of the letters in the word 'INDIA' is 4.

(*c*) For sets A and B, if $n(A)=n(B)$, then $A=B$ is always true.

(*d*) $\{1, 2, 3, 4, 5\}$ and $\{6, 7, 8\}$ are disjoint sets. [*C.B S.E., 1980* (*Delhi*)]

(*e*) For any two sets A and B, $A\subset(A\cup B)$.

(*f*) If A, B and C are three non-empty sets such that $A\cap B=A\cap C$, then $B=C$.

2. If A and B are subsets of a set X, then state which of the following is true or false. Give reasons.

(*a*) $(A\cap B)\subset(A\cup B)$ (*b*) $B\subset(A\cap B)$ [*C.B S.E., 1981* (*A.I.*)]

3. Fill in the blanks so as to make the following statements true.

(*a*) A set which contains no elements is called the......... [*C.B S.E., 1978* (*Delhi*)]

(*b*) When two sets A and B have no elements in common, the sets are said to be......... [*C.B.S.E., 1979* (*A.I.*)]

(*c*) If A and B are any two sets, then $n\ (A\cup B)=$......... [*C.B.S.E., 1982* (*A.I.*)]

(*d*) A'......$A=\phi$, where A' is complement of A. [*C.B.S E., 1984* (*A.I*)]

(*e*) The set V of vowels is a......of the set A of English alphabets.

(*f*) The sets $\{15, 8, 11, 14, 17\}$ and $\{16, 7, 9, 10, 12, 13\}$ are.........

[*C B.S.E., 1977* (*Delhi*)]

4. Which of the following sets is a universal set of the remaining sets :

(*a*) the set of natural numbers (*b*) the set of whole numbers

(*c*) the set of integers (*d*) the set of even integers

(*e*) the set of odd integers.

5. List all the proper subsets of $\{-1, 0, 1\}$

6. If $A\cup B=\phi$, show that $A=\{\ \ \}$ and $B=\phi$.

(Section—B)

7. If $A=\{1, 2, 3, 4, 5\}$, $B=\{1, 2, 4, 6\}$ and $C=(1, 5, 3\}$, write

(*a*) *a* set D such that $D\not\subset B$, $D\subset C$, $D\subset A$

(*b*) *a* set F such that $F\subset B$, $F\not\subset A$, $F\not\subset C$.

8. Given the sets $A=\{1, 9, 11, 15, 17\}$,
$B=\{11, 17, 19, 21\}$, and $C=\{3, 5, 7\}$;
find $(A\cap B)\cup C$.

9. Given $U=\{x : -3\leqslant x\leqslant 3, x\in 1\}$
$A=\{-2, 0, 2\}$ and $B=\{0, 1, 2, 3\}$,
list the elements of the following sets :

(*a*) $U\cap B'$ (*b*) $(A\cup B)'$

10. Given that $A=\{11, 21, 31, 41, \ldots, 91\}$,

$B=\{\text{prime numbers}\}$,

$C=\{7, 14, 21, 35, 42, \ldots\}$.

List the following sets :

(*a*) $A \cap B$ (*b*) $A \cap B'$

(*c*) $(A \cap C) \cup (B \cap C)$

11. If A and B are the subsets of the universal set U, prove the $A \cup B = A \cap B$, if and only if $A=B$.

12. Let $U=\{a, b, c, d, e, f, g\}$ and $A=\{a, b, c, d, e\}$, find $(A \cap A')'$.

(Section—C)

13. Draw a Venn-diagram for three non-empty sets A, B and C so that these sets have the following properties :

$$A \subset B, \qquad A \cap C \neq \phi, \qquad C \not\subset B.$$

14. In a school there are 10 teachers who teach English or Mathematics. Of these, 5 teach English alone and 3 teach English and Mathematics. How many teach Mathematics alone ?

15. In a group of 120 men, 72 can speak Hindi only, and 26 can speak Hindi and English. How many of them can speak English only ?

16. Let A and B be and two sets. If $A = A \cup B$, prove that $\mathbf{B} = \mathbf{A} \cap \mathbf{B}$.

□ □

2

NUMBER SYSTEMS

2.1. NATURAL NUMBERS

We have been using the numbers 1, 2, 3, 4, 5, ... for counting since our childhood. These are called **counting numbers** or **natural numbers**. We denote the set of natural numbers by N.

$$N=\{1, 2, 3, 4, 5, 6, 7, 8, ...\}$$

Natural numbers together with zero are called **whole numbers**.

The set of whole numbers is denoted by W.

$$W=\{0, 1, 2, 3, 4, 5, 6, ...\}$$

The graphic representation of the set of whole numbers is shown below :

N and W, both are *infinite* sets.

We see that N is a *subset* of W.

Properties of W for addition :

(*i*) Closure Property.	If $a, b \in W, a+b \in W$
(*ii*) Commutative Property.	If $a, b \in W, a+b=b+a$
(*iii*) Associative Property.	If $a, b, c \in W, a+(b+c)=(a+b)+c$
(*iv*) Additive Identity.	If $a \in W, a+0=0+a=a$

0 is called the **additive identity** for whole numbers.

Properties of W for Multiplication :

(*i*) Closure Property :	If $a, b \in W, a.b \in W$
(*ii*) Commutative Property :	If $a, b \in W, a.b=b.a$
(*iii*) Associative Property :	If $a, b, c \in W, a.(b.c)=(a.b).c$
(*iv*) Multiplicative Identity :	If $a \in W, 1.a=a.1=a$

1 is called the **multiplicative identity** for whole numbers.

(*v*) Distributive Property of Multiplication over Addition.

If $a, b, c \in W, a.(b+c)=a.b+a\,c$

Subtracting one whole number from another does not always lead to a whole number. So the set of whole numbers is *not* closed for the operation of subtraction. The introduction of *negative numbers e.g* , -1, -2, -3, ... enabled us to remedy of this defect.

2.2. INTEGERS

The set consisting of (*i*) natural numbers, (*ii*) their negatives or additive inverses and (*iii*) zero is called the set of **integers**.

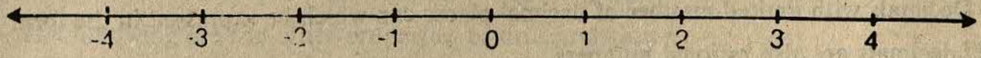
The set of integers is denoted by Z.

The natural numbers 1, 2, 3, 4, 5, ... are called **positive integers**. They are denoted by Z^+.

The negative of natural numbers, -1, -2, -3, -4, ... are called **negative integers**. They are denoted by Z^-.

Zero is called **zero integer**. It is neither positive nor negative.

The graphic representation of the set of integers is shown below :



Note that $N \subset W \subset Z$.

Properties of Z for Addition :

(*i*) Closure Property. If $a, b \in Z$, $a+b \in Z$

(*ii*) Commutative Property. If $a, b \in Z$, $a+b=b+a$

(*iii*) Associative Property. If $a, b, c \in Z$, $a+(b+c)=(a+b)+c$

(*iv*) Additive Identity. If $a \in Z$, $a+0=0+a=a$

0 is called the **additive identity** for integers.

(*v*) Additive Inverse. If $a \in Z$, $a+(-a)=0$, and $(-a) \in Z$

$(-a)$ is called the **additive inverse** of 'a'.

Additive inverse of 'a' is also called **opposite** of 'a'.

Properties of Z for Multiplication :

(*i*) Closure Property : If $a, b \in Z$, $a\,b \in Z$

(*ii*) Commutative Property : If $a, b \in Z$, $a.b=b.a$

(*iii*) Associative Property : If $a, b, c \in Z$, $a.(b.c)=(a.b).c$

(*iv*) Multiplicative Identity : If $a \in Z$, $1.a=a.1=a$

1 is called the **multiplicative identity** for integers.

(*v*) Distributive Property of Multiplication over Addition.

If $a, b, c \in Z$, $a.(b+c)=a.b+a.c$.

Division of one integer by another integer does not always lead to an integer. So the set of integers is *not* closed for the operation of division Such a situation necessitates the introduction of new numbers.

2.3. RATIONAL NUMBERS

The set of integers is enlarged so that there is no restriction on division of integers *except* that of division by zero.

The new extended set of integers is called the *set of rational numbers* and is denoted by Q.

A rational numbers is of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Note that $N \subset W \subset I \subset Q$.

The set of rational number consists of these subsets :

(*i*) The set of **non-negative** rational numbers, such as :

$$\frac{0}{3}, \frac{0}{8}, \frac{4}{9}, \frac{2}{7}, \frac{5}{5}, \frac{8}{1}, \frac{11}{4}, \frac{9}{16}$$

(*ii*) The set of **negative** rational numbers, such as

$$\frac{-3}{1}, \frac{-5}{8}, \frac{-4}{13}, \frac{-14}{7}, \frac{6}{-11}, \frac{7}{-14}, \frac{13}{-16}$$

Note that positive rational numbers are the *same* as fractional numbers.

Decimals with limited number of decimal places can which be expressed in the form $\frac{p}{q}$. So such decimals are also rational numbers.

The operations of addition, subtraction, multiplication and division are always possible in the set of rationals.

Between any two rationals we can always insert at least one rational number. It implies that *there is an infinite number of rational numbers between any two rational numbers.*

Properties of Q for Addition :

(*i*) Closure Property. If $a, b \in Q$, $a+b \in Q$.
(*ii*) Commutative Property. If $a, b \in Q$, $a+b=b+a$
(*iii*) Associative Property. If $a, b, c \in Q$, $a+(b+c)=(a+b)+c$
(*iv*) Additive Identity. If $a \in Q$, $a+0=0+a=a$
0 is called the **additive identity** for rational numbers.
(*v*) Additive Inverse. If $a \in Q$, $a+(-a)=0$, and $(-a) \in Q$
$(-a)$ is called the **additive inverse** of 'a'. Additive inverse of 'a' is also called **opposite**.

Properties of Q for Multiplication :

(*i*) Closure Property : If $a, b \in Q$, $a.b \in Q$.
(*ii*) Commutative Property : If $a, b \in Q$, $a.b=b.a$
(*iii*) Associative Property : If $a, b, c \in Q$, $a.(b.c)=(a.b).c$
(*iv*) Multiplicative Identity : If $a \in Q$, $1.a=a.1=a$
1 is called the **multiplicative identity** for rational numbers.

(*v*) Multiplicative Inverse : If non-zero $a \in Q$, $a.\frac{1}{a}=\frac{1}{a}.a=1$, where $\frac{1}{a} \in Q$.
$\frac{1}{a}$ is called the **multiplicative inverse** of a.

(*vi*) Distributive Property of Multiplication over Addition :
If $a, b, c \in Q$, $a.(b+c)=a.b+a.c$.

Two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ are said to be **equal** to each other, if and only if $ps=qr$.

We can state it as under :

$$\frac{p}{q}=\frac{r}{s}, \text{ if and only if } ps-rq=0$$

$$\frac{p}{q}>\frac{r}{s}, \text{ if and only if } ps-rq>0,$$

and $$\frac{p}{q}<\frac{r}{s}, \text{ if and only if } ps-rq<0.$$

It is now clear that given two rational numbers a and b, *only one* of the following three possibilities will hold good

either $a=b$ or $a>b$ or $a<b$

The following are the other properties of order relation :

(*i*) If $a, b, c \in Q$, and $a>b$, $b>c$, then $a>c$

(*ii*) If $a, b, c \in Q$, and $a>b$, then $a+c>b+c$

(*iii*) If $a, b, c \in Q$, and $a>b$, $c>0$, then $ac>bc$.

We know how to represent rational numbers on a number line. Rational numbers are arranged in order on the number line.

Every rational number is represented by one and only one point on the number line.

The system $\{Q, +, .\}$ is said to be a **field** because of the eleven properties listed above. The rational numbers under operations of addition and multiplication form a field.

EXERCISE—2 (a)

(Section—A)

1. Can you always divide one whole number by another whole number and get a whole number quotient ?

   Illustrate your answer with examples.

2. (*a*) Is the set of even whole numbers closed under addition ?

   (*b*) Is the set of odd whole numbers closed under addition ?

3. (*a*) Is the set of even whole numbers closed under multiplication ?

   (*b*) Is the set of odd whole numbers closed under multiplication ?

4. (*a*) Is the set of natural numbers finite ?

   (*b*) Is the set of whole numbers infinite ?

   (*c*) Is the set of integers finite ?

5. What is the opposite of 0 ?

6. (*a*) Is every natural number a rational number ?

   (*b*) Is every whole number a rational number ?

   (*c*) Is every integer a rational number ?

7. Which two rational numbers are their own multiplicative inverses ?

8. Find the integral value of x, for which $\frac{5}{x}$ is an integer.

9. Find the integral value of x, for which $\frac{7}{x}$ is a natural number.

10. What is the number midway between $-\frac{1}{2}$ and $\frac{3}{2}$?

11. Can you find three different rational numbers between any two rational numbers ?

(Section—B)

12. Arrange in the ascending order :

$$\frac{2}{3}, \frac{4}{7}, \frac{5}{11}.$$

13. Write the following numbers in descending order :

$$0{\cdot}23, \frac{7}{25}, \frac{5}{18}.$$

14. Find the additive inverse of (*a*) $\frac{-4}{7}$, (*b*) $\frac{-5}{8}$.

15. Find the multiplicative inverse of (*a*) $\frac{-8}{13}$ (*b*) $\frac{9}{-16}$.

(Section—C)

16. If a and b are two rational numbers, prove that $a+b$, $a-b$, ab are rational numbers. If $b \neq 0$, show that $\frac{a}{b}$ is also a rational number.

2.4. IRRATIONAL NUMBERS

Let us now consider the representation of rational numbers as decimals.

$$\frac{7}{10}=0{\cdot}7\ ; \quad \frac{38}{100}=0{\cdot}38\ ; \quad \frac{489}{100}=4{\cdot}89.$$

$$\frac{3}{25}=\frac{3\times4}{25\times4}=\frac{12}{100}=0{\cdot}12\ ; \quad \frac{7}{125}=\frac{7\times8}{125\times8}=\frac{56}{1000}=0{\cdot}056.$$

We may also use the long division as follows :

$$\begin{array}{r} 0{\cdot}875 \\ 8\overline{)7{\cdot}000} \\ 64 \\ \hline 60 \\ 56 \\ \hline 40 \\ 40 \\ \hline \times \end{array} \qquad \text{So } \frac{7}{8}=0{\cdot}875$$

$$\begin{array}{r} 1{\cdot}425 \\ 40\overline{)57{\cdot}000} \\ 40 \\ \hline 170 \\ 160 \\ \hline 100 \\ 80 \\ \hline 200 \\ 200 \\ \hline \times \end{array} \qquad \text{So } \frac{57}{40}=1{\cdot}425$$

The above rational numbers, when written in decimal form, are terminating decimals. We may, however, consider some other rational numbers such as $\frac{1}{3}$, $\frac{3}{11}$, etc.

$$\begin{array}{r} 0\cdot333 \\ 3\overline{)1\cdot000} \\ 9 \\ \hline 10 \\ 9 \\ \hline 10 \\ 9 \\ \hline 1 \end{array}$$

Here the division does not end or terminate. At every stage we get the same remainder 1.

$$\frac{1}{3}=0\cdot333\ldots$$

We also write it as $\frac{1}{3}=0\cdot\dot{3}$

The '·' over 3 indicates that 3 keeps on repeating without end.

$$\begin{array}{r} 0\cdot2727 \\ 11\overline{)3\cdot0000} \\ 22 \\ \hline 80 \\ 77 \\ \hline 30 \\ 22 \\ \hline 80 \\ 77 \\ \hline 3 \end{array}$$

Sometimes we also put a '—' over digits which repeat.

$$\frac{3}{11}=0\cdot272727\ldots$$

We also write it as $\frac{3}{11}=0\cdot\dot{2}\dot{7}$ or $0\cdot\overline{27}$

The digits '27' keep on repeating without end. So we put '··' over them.

These rational numbers are expressed as non-terminating repeating or periodic decimals. Every terminating decimal can be considered as non-terminating since 0·7 can be written as 0·700000 .., 0 is being repeated.

A rational number is a repeating or periodic decimal. (It may terminate or not terminate).

We also know that every periodic decimal can be expressed as a rational number. Thus

$$0\cdot7=\frac{7}{9}\ ; \qquad 0\cdot\dot{1}\dot{9}=\frac{19}{99}\ ; \qquad 0\cdot\dot{3}\dot{5}=\frac{35}{99}.$$

Let $0\cdot\dot{7}=0\cdot777\ldots=q$

Then $10q=7\cdot777$

Subtracting we get,

$9q=7$ So $q=\frac{7}{9}$

Let $0\cdot35=0\cdot3535\ldots=q$

Then $100q=35\cdot3535\ldots$

Subtracting we get

$99q=35$ So $q=\frac{35}{99}$

We now consider some decimals which are not periodic (and do not terminate).

(*i*) 3·14159... (*ii*) 1·4142135...

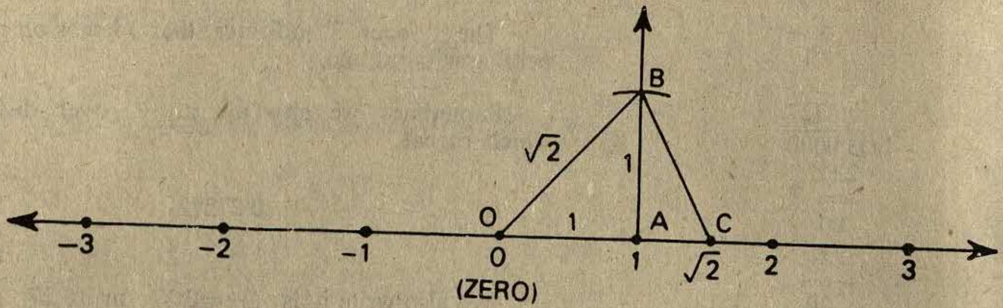
None of these decimals can be expressed as a rational number. Such numbers are called *irrational numbers*.

An irrational number is a non-repeating (or non-periodic) and non-terminating decimal. We shall now consider irrational number in a different way.

We have taken for granted that *every point on the number-line represents a number.* We have observed that *corresponding to every rational number, there is a point on the number-line.*

Is every number represented by a point on the number-line, a rational number ? Take the number-line.

Let point O represent number 0 and point A represent number 1.



At A, draw $\overrightarrow{AB}$ perpendicular to the number-line. On $\overrightarrow{AB}$, cut off $\overline{AB}$ such that $AB=1$. Draw segment OB.

With centre O and radius OB, draw an arc intersecting the number-line at C.

Then $OC=OB$.

By Pythagoras theorem, we known

$$OB^2=OA^2+AB^2$$

i.e., $$OB^2=1^2+1^2$$

i.e., $$OB^2=2$$

So OB represents a *positive* number such that its square is 2.

Let us denote that positive number by $\sqrt{2}$.

$\sqrt{2}$ is called a **square root** of 2.

Therefore $$OB=\sqrt{2}$$

Then $$OC=\sqrt{2}$$

Thus C represents number $\sqrt{2}$ on the number-line.

Is $\sqrt{2}$ a rational number ?

Can $\sqrt{2}$ be represented as a fraction $\frac{a}{b}$, where a and b are integers and $b\neq 0$?

In order to show that $\sqrt{2}$ is not a rational number, we shall make use of three simple but important ideas.

(1) **Indirect Reasoning.** We shall assume that $\sqrt{2}$ is a rational number and then show that this assumption leads to an impossible conclusion.

(2) An even number is a multiple of 2.

(3) If the square of a number is even, then the number is an even number.

Proof. If possible let $\sqrt{2}$ be a rational number.

Then $\sqrt{2}=\frac{p}{q}$, where p and q are integers prime to each other and $q \neq 0$.

Squaring, we get $2=\frac{p^2}{q^2}$

$$2q^2=p^2.$$

Since p and q are integers, p^2 and q^2 are also integers. So p^2 is an even integer.

Thus p is also an even integer.

Let $p=2k$, where k is an integer.

$$2q^2=(2k^2)$$
$$=4k^2$$

or $$q^2=2k^2$$

Thus q^2 is an even integer and, therefore, q is also an even integer.

Hence we find that p and q both are even integers. Then p and q have a common factor 2. This contradicts our supposition.

Thus $\sqrt{2}$ is not a rational number.

We can easily find that there are infinitely many numbers which are not rational numbers, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, ... In general $\sqrt{n}$ where n is a positive integer and a nonperfect square. Similarly the cube root of any integer is a irrational number unless the integer is a perfect cube. The number π is also an irrational number.

Let us find out whether $\sqrt{2}$ has a decimal representation, and if so, examine the nature of the decimal representation.

Let us consider $\sqrt{2}$. $\sqrt{2}$ is a positive number such that its square is 2. The process of finding the square root of 2 by division method can give us the decimal representation of $\sqrt{2}$. We, will, however follow here a more elementary method.

If $n=\sqrt{2}$, then $n^2=2$

Since $1^2=1$, and $2^2=4$ we can say that $\sqrt{2}$ is greater than 1 and less than 2. That is $1<\sqrt{2}<2$.

We can get closer approximations if we work with tenths

$$1{\cdot}1^2=1{\cdot}21$$
$$1{\cdot}2^2=1{\cdot}44$$
$$1{\cdot}3^2=1{\cdot}69$$
$$\left.\begin{array}{l}1{\cdot}4^2=1{\cdot}96\leftarrow\\ 1{\cdot}5^2=2{\cdot}25\leftarrow\end{array}\right|$$

Thus $1{\cdot}4<\sqrt{2}<1{\cdot}5$.

This process may be continued by actually calculating squares.

$$1{\cdot}41<\sqrt{2}<1{\cdot}42$$
$$1{\cdot}414<\sqrt{2}<1{\cdot}415$$
$$1{\cdot}4142<\sqrt{2}<1{\cdot}4143$$

Using seven place decimals, we have

$$1{\cdot}4142135<\sqrt{2}<1{\cdot}4142136.$$

Proceeding in the same way we can obtain decimal representations for $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, etc.

$\sqrt{3}=1{\cdot}7320508$ .., $\sqrt{5}=2{\cdot}2360680$. $\sqrt{7}=2{\cdot}6457513$ ..

We can use decimal approximation to locate $\sqrt{2}$ and other irrationals on number-line.

Example 1. State whether the given number is rational or irrational.

(*a*) $-\frac{5}{9}$ (*b*) $\sqrt{31}$ (*c*) $\sqrt[3]{5}$ (*d*) 2·878787....

Solution. (*a*) $\frac{-5}{9}$ is of the form $\frac{p}{q}$, where p and q are integers but $q \neq 0$.

So it is a rational number.

(*b*) $\sqrt{31}$ is the square root of a number which is not a perfect square of a rational number. So it is an irrational number

(*c*) $\sqrt[3]{5}$ is the cube root of a number which is not a perfect cube of a rational number. So it is an irrational number.

(*d*) 2·878787... is a non-terminating but repeating decimal. So it is a rational number.

Example 2. Determine a rational and an irrational number between $\frac{5}{8}$ and $\frac{2}{3}$

Solution. Let us express the given numbers as decimals :

$\frac{5}{8} = 0{\cdot}625$ and $\frac{2}{3} = 0{\cdot}6666..$ *i.e.*, $0{\cdot}6\dot{6}$.

So we can write a terminating decimal for a rational number between these decimals, *e.g.*, 0·64.

Here $0{\cdot}64 > 0{\cdot}625$ and $0{\cdot}64 < 0{\cdot}6\dot{6}$.

Hence 0·64 can be taken as a rational number between $\frac{5}{8}$ and $\frac{2}{3}$.

Similarly, we can write a non-terminating but non-repeating decimal for an irrational number between 0·625 and $0{\cdot}6\dot{6}$.

Let us take 0·64312703...as the required number.

Here $0{\cdot}64312703... > 0{\cdot}625$ and $0{\cdot}64312703... < 0{\cdot}6\dot{6}$.

Hence 0·64312703.. can be taken as an irrational number between $\frac{5}{8}$ and $\frac{2}{3}$

Let us consider sum and product of two irrational numbers.

(*i*) If a is irrational, then $-a$ is also irrational.

For example, $\sqrt{2}$ is an irrational number. So $-\sqrt{2}$ is also an irrational number.

(*ii*) **The sum of a rational number and an irrational number is always irrational.**

For example, 5 is a rational number and $\sqrt{3}$ is an irrational number. So $5+\sqrt{3}$ is an irrational number.

(*iii*) **The product a non-zero rational number and an irrational number is always irrational.**

For example, 7 is a non-zero rational number and $\sqrt{5}$ is an irrational number. So $7 \times \sqrt{5}$ *i.e*, $7\sqrt{5}$ is an irrational number.

We know that $\sqrt{3}$ is an irrational number. Hence $-\sqrt{3}$ is also an irrational number. Their sum is $\sqrt{3}+(-\sqrt{3})$ *i.e.*, 0 which is a rational number.

We know that $\sqrt{2}$ and $\sqrt{3}$ are two irrational numbers.

Their sum is $\sqrt{2}+\sqrt{3}$. Is it rational or irrational ?

Let $\sqrt{2}+\sqrt{3}$ be a rational number.

The square of a rational number is also a rational number.

So, $(\sqrt{2}+\sqrt{3})^2$ should be rational.

$\Rightarrow$ $2+3+2\sqrt{2}\sqrt{3}$ should be rational.

$\Rightarrow$ $5+2\sqrt{6}$ is a rational number. But $\sqrt{6}$ is an irrational number. Since the sum of a rational number and an irrational number is always irrational, $5+2\sqrt{6}$ is irrational.

Hence $(\sqrt{2}+\sqrt{3})^2$ is irrational. This contradiction proves that $\sqrt{2}+\sqrt{3}$ is not rational *i.e.*, $\sqrt{2}+\sqrt{3}$ is an irrational number.

What do you infer ?

The sum of two irrational numbers may be rational in some cases and irrational in some other cases.

We know that $\sqrt{2}$ and $\sqrt{3}$ are two irrational numbers.

Their product $=\sqrt{2}\times\sqrt{3}=\sqrt{6}$ which is an irrational number because 6 is not a perfect square.

Again $\sqrt{5}$ and $\sqrt{20}$ are two irrational numbers.

$$\text{Their product}=\sqrt{5}\times\sqrt{20}=\sqrt{5\times 20}$$

$$=\sqrt{100}=10 \text{ which is a rational number.}$$

What do you infer ?

The product of two irrational numbers may be rational in some cases and irrational in some other cases.

We are familiar with the number π. *The ratio of the length of circumference of a circle to the length of its diameter is always constant.* This constant is an irrational number. It is denoted by the Greek letter. π.

Since π is irrational, its decimal representation is non-terminating and non-repeating.

The value of π to a few places of decimals is given by

$$\pi=3{\cdot}14159265\ldots$$

The rational number $\frac{22}{7}$ is very often taken to be an approximate value of π. We should note that $\frac{22}{7}$ is not equal to π and it approximates π to two places of decimals.

EXERCISE—2 (*b*)

(Section—A)

1. State which of the following numbers are irrational :

(*a*) $\sqrt{7}$ (*b*) $\sqrt{12}$ (*c*) $\sqrt[3]{3}$ (*d*) $\sqrt{36}$ (*e*) $\sqrt[3]{10}$

(*f*) $0{\cdot}57390682\ldots$ (*g*) $0{\cdot}\overline{714285}$ (*h*) $5{\cdot}74625$ (*i*) $1{\cdot}23414141\ldots$

2. Write down two irrational numbers between $1{\cdot}5$ and $1{\cdot}6$.

3. Write down three irrational numbers between $2{\cdot}3$ and $3{\cdot}2$.

4. Show $\sqrt{5}$ on the number line.

5. On a number line, locate the point corresponding to $-\sqrt{3}$.

Example 1. State whether the given number is rational or irrational.

(*a*) $-\frac{5}{9}$ (*b*) $\sqrt{31}$ (*c*) $\sqrt[3]{5}$ (*d*) 2·878787....

Solution. (*a*) $\frac{-5}{9}$ is of the form $\frac{p}{q}$, where p and q are integers but $q \neq 0$.

So it is a rational number.

(*b*) $\sqrt{31}$ is the square root of a number which is not a perfect square of a rational number . So it is an irrational number

(*c*) $\sqrt[3]{5}$ is the cube root of a number which is not a perfect cube of a rational number. So it is an irrational number.

(*d*) 2·878787... is a non-terminating but repeating decimal. So it is a rational number.

Example 2. Determine a rational and an irrational number between $\frac{5}{8}$ and $\frac{2}{3}$

Solution. Let us express the given numbers as decimals :

$$\frac{5}{8} = 0{\cdot}625 \text{ and } \frac{2}{3} = 0{\cdot}6666.. \text{ i.e., } 0{\cdot}6\dot{6}.$$

So we can write a terminating decimal for a rational number between these decimals, *e.g.*, 0·64.

Here $0{\cdot}64 > 0{\cdot}625$ and $0{\cdot}64 < 0{\cdot}6\dot{6}$.

Hence 0·64 can be taken as a rational number between $\frac{5}{8}$ and $\frac{2}{3}$.

Similarly, we can write a non-terminating but non-repeating decimal for an irrational number between 0·625 and $0{\cdot}6\dot{6}$.

Let us take 0·64312703...as the required number.

Here $0{\cdot}64312703... > 0{\cdot}625$ and $0{\cdot}64312703... < 0{\cdot}6\dot{6}$.

Hence 0·64312703.. can be taken as an irrational number between $\frac{5}{8}$ and $\frac{2}{3}$

Let us consider sum and product of two irrational numbers.

(*i*) If a is irrational, then $-a$ is also irrational.

For example, $\sqrt{2}$ is an irrational number. So $-\sqrt{2}$ is also an irrational number.

(*ii*) **The sum of a rational number and an irrational number is always irrational.**

For example, 5 is a rational number and $\sqrt{3}$ is an irrational number. So $5+\sqrt{3}$ is an irrational number.

(*iii*) **The product a non-zero rational number and an irrational number is always irrational.**

For example, 7 is a non-zero rational number and $\sqrt{5}$ is an irrational number. So $7 \times \sqrt{5}$ *i.e*, $7\sqrt{5}$ is an irrational number.

We know that $\sqrt{3}$ is an irrational number. Hence $-\sqrt{3}$ is also an irrational number. Their sum is $\sqrt{3}+(-\sqrt{3})$ *i.e.*, 0 which is a rational number.

We know that $\sqrt{2}$ and $\sqrt{3}$ are two irrational numbers.

Their sum is $\sqrt{2}+\sqrt{3}$. Is it rational or irrational ?

Let $\sqrt{2}+\sqrt{3}$ be a rational number.

The square of a rational number is also a rational number.

So, $(\sqrt{2}+\sqrt{3})^2$ should be rational.

$\Rightarrow$ $2+3+2\sqrt{2}\sqrt{3}$ should be rational.

$\Rightarrow$ $5+2\sqrt{6}$ is a rational number. But $\sqrt{6}$ is an irrational number. Since the sum of a rational number and an irrational number is always irrational, $5+2\sqrt{6}$ is irrational.

Hence $(\sqrt{2}+\sqrt{3})^2$ is irrational. This contradiction proves that $\sqrt{2}+\sqrt{3}$ is not rational *i.e.*, $\sqrt{2}+\sqrt{3}$ is an irrational number.

What do you infer ?

The sum of two irrational numbers may be rational in some cases and irrational in some other cases.

We know that $\sqrt{2}$ and $\sqrt{3}$ are two irrational numbers.

Their product $=\sqrt{2}\times\sqrt{3}=\sqrt{6}$ which is an irrational number because 6 is not a perfect square.

Again $\sqrt{5}$ and $\sqrt{20}$ are two irrational numbers.

Their product $=\sqrt{5}\times\sqrt{20}=\sqrt{5\times 20}$

$=\sqrt{100}=10$ which is a rational number.

What do you infer ?

The product of two irrational numbers may be rational in some cases and irrational in some other cases.

We are familiar with the number π. *The ratio of the length of circumference of a circle to the length of its diameter is always constant.* This constant is an irrational number. It is denoted by the Greek letter. π.

Since π is irrational, its decimal representation is non-terminating and non-repeating.

The value of π to a few places of decimals is given by

$$\pi=3{\cdot}14159265\ldots$$

The rational number $\frac{22}{7}$ is very often taken to be an approximate value of π. We should note that $\frac{22}{7}$ is not equal to π and it approximates π to two places of decimals.

EXERCISE—2 (*b*)

(Section—A)

1. State which of the following numbers are irrational :

(*a*) $\sqrt{7}$ (*b*) $\sqrt{12}$ (*c*) $\sqrt[3]{3}$ (*d*) $\sqrt{36}$ (*e*) $\sqrt[3]{10}$

(*f*) $0{\cdot}57390682\ldots$ (*g*) $0{\cdot}\overline{714285}$ (*h*) $5{\cdot}74625$ (*i*) $1{\cdot}23414141\ldots$

2. Write down two irrational numbers between $1{\cdot}5$ and $1{\cdot}6$.

3. Write down three irrational numbers between $2{\cdot}3$ and $3{\cdot}2$.

4. Show $\sqrt{5}$ on the number line.

5. On a number line, locate the point corresponding to $-\sqrt{3}$.

(Section—B)

6. Find an irrational number between $\frac{2}{5}$ and $\frac{3}{7}$.

7. Write two irrational numbers between $\frac{1}{2}$ and $\frac{1}{3}$.

8. Determine a rational and an irrational numbers between $\frac{5}{8}$ and $\frac{5}{7}$

9. Prove that $\sqrt{3}$ is an irrational number.

10. Show that $\sqrt{7}$ is an irrational number.

11. Prove that $\sqrt{3}-\sqrt{2}$ is irrational.

12. Write two irrational numbers whose sum is
    (*a*) a rational number (*b*) an irrational number.

13. Write two irrational numbers whose product is
    (*a*) a rational number (*b*) an irrational number.

14. Write two irrational numbers whose difference is
    (*a*) a rational number (*b*) an irrational number.

15. Write two irrational numbers whose quotient is
    (*a*) a rational number (*b*) an irrational number.

2.5. REAL NUMBERS

A rational number is a terminating or non-terminating but repeating decimal.

Numbers like $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc., *cannot* be expressed as a quotient of two integers. Such numbers are called *irrational numbers.*

An irrational number is a non-terminating and non-repeating decimal.

The set consisting of the union of the rational numbers and irrational numbers is called the set of *real* numbers.

The set of real numbers is denoted by 'R'. If S represents the set of irrational numbers, then

$$Q \cup S = R.$$

No rational number is irrational and no irrational number is ration al. Therefore such sets are *disjoint.*

$\therefore\ Q \cup S = \phi$ Also $Q \subset R$ and $S \subset R$.

Note that $N \subset W \subset Z \subset Q \subset R$.

Properties of R for Addition

(*i*) Closure Property. If $a, b \in R$, $a+b \in R$

(*ii*) Commutative Property. If $a, b \in R$, $a+b=b+a$

(*iii*) Associative Property. If $a, b, c \in R$, $(a+b)+c=a+(b+c)$

(*iv*) Additive Identity. If $a \in R$, $a+0=0+a=a$

0 is called the **additive identity** for real numbers.

(*v*) Additive Inverse. If $a \in R$, $a+(-a)=0$ and $(-a) \in R$

$-a$ is called the **additive inverse** of 'a'. Additive inverse of 'a' is also called **oppositive.**

Properties of R for Multiplication

(*i*) Closure Property. If $a, b \in R$, $a \cdot b \in R$

(*ii*) Commutative Property. If $a, b \in R$, $a \cdot b = b \cdot a$

(*iii*) Associative Property. If $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(*iv*) Multiplicative Identity. If $a \in R$, $1 \cdot a = a \cdot 1 = a$

1 is called the **multiplicative identity** for real numbers.

(*v*) Multiplicative Inverse. If non-zero $a \in R$, $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$, where $\frac{1}{a} \in R$

$\frac{1}{a}$ is called the **multiplicative inverse** of a.

(*vi*) Distributive Property of Multiplication over Addition.

$$\text{If } a, b, c \in R,\ a \cdot (b+c) = a \cdot b + a \cdot c$$

Thus, the set of real numbers under operations of addition and multiplication form a **field.**

Let a be a real number (*rational* or *irrational*).

Now there is a point on the number line which corresponds to the number a. The distance of this point from the point 0 (zero mark) is called the **absolute value** of a.

Since distance is also *non-negative*, the absolute value of a real number a is always *positive* and is written as $|a|$.

Thus $\quad |3| = 3 \quad$ and $\quad |-3| = 3$

$$\left|-\frac{4}{9}\right| = \frac{4}{9} \quad \text{and} \quad \left|\frac{4}{9}\right| = \frac{4}{9}$$

We can also say that

$$|a| = a, \text{ if } a \geqslant 0 \quad \text{and} \quad |a| = -a, \text{ if } a < 0.$$

Thus $\quad |5| = 5$, since $5 > 0$

and $\quad |-5| = -(-5)$ *i.e.*, 5, since $-5 < 0$.

Note that *absolute value of zero is zero.*

Example 1. Evaluate :

(*a*) $|5| - |-3| + |-7| - |2|$

(*b*) $|(-7)\times(-2)| + |-5| - |-4|$.

Solution. (*a*)
$$\begin{aligned} |5| - |-3| + |-7| - |2| &= 5-3+7-2 \\ &= 12-5 \\ &= 7. \end{aligned}$$

(*b*)
$$\begin{aligned} |(-7)\times(-2)| + |-5| - |-4| &= |14| + 5 - 4 \\ &= 14+5-4 \\ &= 19-4 \\ &= 15. \end{aligned}$$

Example 2. Find the value of x, when $|x-3| = 8$.

Solution. When $x-3 \geqslant 0$, $|x-3| = 8$ gives $x-3 = 8$

$\Rightarrow \quad x = 8+3 \quad$ *i.e.*, $\quad x = 11$

When $x-3 < 0$, $|x-3| = 8$ gives $-(x-3) = 8$

$\Rightarrow \quad -x+3=8$

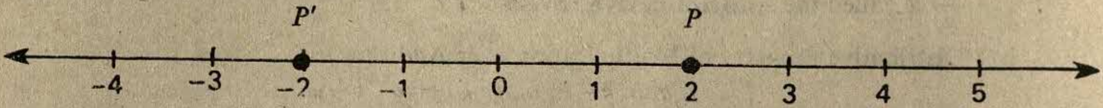
$\Rightarrow \quad -x=5 \quad$ *i.e.*, $\quad x=-5.$

Hence $\quad x=-5, 11.$

Example 3. Find all the real numbers x on the number line which satisfy the following equality/inequality :

(*a*) $|x|=2$ (*b*) $|x| \geqslant 2$ (*c*) $|x| < 2.$

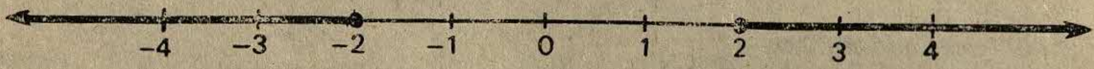
Solution. (*a*) If $x \geqslant 0$, $|x|=2$ gives $x=2$. It is represented by point P on the number line given below :



If $x<0$, $|x|=2$ gives $-x=2 \Rightarrow x=-2$. It is represented by point P' on the number line given above.

(*b*) When $x \geqslant 0$, $|x| \geqslant 2$ gives $x \geqslant 2$. We mark point P on the number line corresponding to number 2. Then point P and all the points on the number line starting P to its right represent the inequality.

When $x<0$, $|x| \geqslant 2$ gives $-x \geqslant 2$. $\Rightarrow x \leqslant -2$. We mark point P' on the number line corresponding to number -2. Then point P' and all the points on the number line starting from P' to its left represent the inequality. Therefore graph of these points is as shown below :



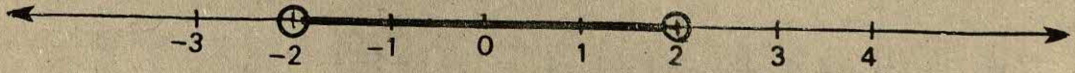
(*c*) If $x \geqslant 0$, $|x| < 2$ gives $x<2$.

This means all the points on the number line lying between 0 and 2, including 0 but excluding 2.

If $x<0$, $|x|<2$ gives $-x<2 \Rightarrow x>-2$.

This means all the points on the number line lying between -2 and 0, excluding -2 but including 0.

Combining these two, the inequality will represent all the points on the number line lying between -2 and 2, excluding the points -2 and 2. Its graph is shown below :



Example 4. If $x * y = |x-y|$, find $5 * 3$ and $(-5) * 3$. [*C.B.S.E., 1984 (Delhi)*]

Solution. Given $\quad x * y = |x-y|$.

Then $\quad 5 * 3 = |5-3|$

$= |2| = 2.$

Also $\quad (-5) * 3 = |-5-3|$

$= |-8| = 8.$

Learn the following results by heart. However, you verify them by assigning different real values to x and y:

(*i*) $x \leqslant |x|$ (*ii*) $|x| = |-x|$

(*iii*) $|x+y| \leqslant |x| + |y|$ (*iv*) $|x-y| \geqslant |x| - |y|$

(*v*) $|x \cdot y| = |x| \cdot |y|$ (*vi*) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, where $y \neq 0$.

EXERCISE—2 (c)

(Section—A)

1. List four distinct real numbers.

2. (*a*) Is every rational number a real number ?
(*b*) Is every real number a rational number ?

3. (*a*) Is every integer a real number.
(*b*) Is every real number an integer ?

4. Find
(*a*) $R \cap Q$ (*b*) $R \cup S$ (*c*) $R \cap S$
(*d*) $Q \cup S$ (*e*) $Q \cap S$ (*f*) $R \cup Q$.

5. Is there a real number that is not rational ?

6. Is there a rational number that is not real ?

(Section—B)

7. (*a*) Write a real number which is not rational ?
(*b*) Write a real number which is not irrational ?

8. Let $A = \left\{5, \frac{2}{7}, \pi, \sqrt{7}, -\frac{\pi}{2}, -5, 2+\sqrt{3}, 3\right\}$, find the subsets of A containing elements which are
(*a*) natural numbers, (*b*) integers,
(*c*) rational numbers, (*d*) irrational numbers,
(*e*) real numbers.

9. If the universal set is the set of real number, find the complement of
(*a*) the set of rational numbers,
(*b*) the set of irrational numbers,
(*c*) the set of positive real numbers,
(*d*) the set of negative numbers,
(*e*) the set of non-positive numbers,
(*f*) the set of integers ending in zero.

10. Evaluate
(*a*) $18 - |-7| + |11| - |-22|$
(*b*) $|(-4)\times(-2)| - |-6| + |(-5)\times 3|$
(*c*) $\left|\frac{13}{2}\right| + (3-\sqrt{2}) - |\sqrt{2}| - \left|-\frac{1}{2}\right|$
(*d*) $|(-4)\times(-2)| + |-4| - |-2| + |-\sqrt{3}|$.

11. Find the value of x, when

(a) $|2-x|=5$ (b) $|3x-2|=4$.

12. Find all the real numbers x on the number line which satisfy the following equality/inequality

(a) $|x|=3$, (b) $|x|\geqslant 3$
(c) $|x|<3$.

13. Find the points x on the number line such that

(a) $|x-5|<3$ (b) $|x-5|=3$
(c) $|x-5|>3$.

14. Draw the graphs of the following equality and inequality on the number line :

(a) $|x|=\sqrt{5}$ (b) $|x|\geqslant\sqrt{5}$
(c) $|x|<\sqrt{5}$.

15. By assigning any values to a and b where a and $b\in R$, verify that

(a) $|a+b|\leqslant|a|+|b|$
(b) $|a\cdot b|=|a|\cdot|b|$
(c) $|a-b|\geqslant|a|-|b|$
(d) $|a-b|\geqslant|b|-|a|$
(e) $|a|+|b|\geqslant|a-b|$
(f) $\left|\frac{a}{b}\right|=\frac{|a|}{|b|}$, $b\neq 0$.

REVIEW EXERCISE—II

(Section—A)

1. Fill in the blanks to make the following statements true :

(a) $|-5|-|3|=\ldots$ [C.B.S.E., 1981 (A.I.)]

(b) If $|x+2|=3$, then $x=$......or...... [C.B.S.E., 1982 (A.I.)]

(c) Every number of the form $\frac{p}{q}$, where p and q are integers and $q\neq 0$, is a...number.

(d) If the square of a natural number is even, the number itself is ......

(e) If x and y are positive real numbers and $x * y$ is represented by a positive real number $\sqrt{xy}$, then $6 * (3 * 12)$ represents the number.........

[C.B.S.E, 1977 (Delhi)]

2. (a) Write the least positive integer.

(b) Write the least non-negative integer.

(c) Write the greatest negative integer.

(d) Write the greatest non-negative integer.

3. Express $\frac{3}{8}$ as a decimal fraction. [C.B.S.E., 1979 (Delhi)]

4. When is a number d said to be additive inverse of another number b ?

5. Find the value of $|-x|-|y|$, when $x=-2$, $y=-3$. [C.B.S.E., 1984 (A.I.)]

6. Give a counter example to show that $\frac{b}{a} \Rightarrow a \geqslant b$ is not always true in the set of integers.

7. When x and y are positive real numbers, let $x * y$ denote the positive real number $\sqrt{x(y+1)}$. Find the value of $3 * 11$ [*C.B.S.E., 1978* (*Delhi*)]

8. Express $6{\cdot}\overline{46}$ in the form $\frac{p}{q}$ where p, q are integers, $q \neq 0$. [*C.B.S.E., 1983* (*Delhi*)]

9. Find two rational numbers between $\frac{4}{5}$ and $\frac{7}{13}$. [*C.B.S.E., 1984* (*A.I.*)]

(Section—B)

10. Find the value of x in $|2-x|=5$.

11. Prove that $\sqrt{2}+\sqrt{3}$ is an irrational number.

(Section—C)

12. The sum of two given real numbers is a rational number. Show that the numbers are either both rational or both irrational.

□ □

3

SURDS

3.1. SURD

We know that $\sqrt{3}, \sqrt[3]{7}, \pi, 1{\cdot}4142135\ldots, 4\sqrt{20}, 0{\cdot}131331333\ldots$ are irrational numbers.

We know that there is no rational number whose square is 2. However, there is a real number whose square is 2. In decimal form this real number is written as $1{\cdot}4142135\ldots$ which is a non-terminating and non-repeating decimal *i.e.*, irrational.

We write this real number in short form by the symbol $\sqrt{2}$ or $2^{1/2}$.

Similarly, the irrational (real) number whose square is 3, is denoted by the symbol $\sqrt{3}$ or $3^{1/2}$.

Also, the irrational (real) number whose cube is 2, is denoted by the symbol $\sqrt[3]{2}$ or $2^{1/3}$.

Observe the following real (irrational) numbers : $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt[3]{2}, \sqrt[3]{5}, \sqrt[4]{8}, \sqrt[5]{10}$.

Such numbers are called **surds** or **radicals.**

If a is a positive rational number, which cannot be expressed as the nth power of some rational number, then the irrational number $\sqrt[n]{a}$ or $a^{1/n}$ *i.e.*, the positive nth root of a is called a **surd** or a **radical**.

The symbol $\sqrt[n]{\ }$ is called the **radical sign**, n is called the **order** of the surd and a is called the **radicand**.

We should remember that

(*i*) $\sqrt[n]{a}$ is a surd,

(*ii*) a is a rational number, and

(*iii*) $\sqrt[n]{a}$ is an irrational number.

For example, $\sqrt{25}$ is *not* a surd because $25=5^2$. Then $\sqrt{25}$ equals to 5 which is a rational number.

Similarly $\sqrt{2+\sqrt{7}}$ is an irrational number but it is *not* a surd because the number $2+\sqrt{7}$ is not a rational number.

Also $4\sqrt{3}$ is a surd, but the same written as $\sqrt{\sqrt{3}}$ is *not* considered a surd because $\sqrt{3}$ is not a rational number.

3.2. ORDER OF SURDS

The order of a surd is determined by its index. $\sqrt{3}$ is a surd of **second order,** $\sqrt[3]{4}$ is a surd of **third order,** $\sqrt[4]{3}$ is a surd of **fourth order.**

Observe the following surds :

$$\sqrt{5},\ \sqrt{7},\ \sqrt{11},\ \sqrt{31},\ \sqrt{52},\ \sqrt{75}.$$

Each of them is a surd of second order *i.e.*, all of them are **quadratic** surds.

$\sqrt[3]{4}$, $\sqrt[3]{9}$, $\sqrt[3]{28}$, $\sqrt[3]{60}$ are surds of third order. All of them are **cubic** surds.

3.3. PURE AND MIXED SURDS

A surd which consists only of an irrational number is called a pure surd.

For example, $\sqrt{2}$, $\sqrt{8}$, $\sqrt[3]{15}$, $\sqrt[4]{25}$, $\sqrt[5]{100}$ are pure surds.

A surd consisting of the product of a rational number and an irrational number is called a mixed surd.

For example, $2\sqrt{3}$, $5\sqrt{2}$, $4\sqrt[3]{6}$, $8\sqrt[4]{10}$ are mixed surds.

Now
$$\begin{aligned}3\sqrt{2}&=3\times2^{1/2}=(3^2)^{1/2}\times2^{1/2}\\&=9^{1/2}\times2^{1/2}=(9\times2)^{1/2}\\&=(18)^{1/2}=\sqrt{18}\end{aligned}$$

Thus we can express a mixed surd as a pure surd.

Also
$$\begin{aligned}\sqrt{50}&=\sqrt{25\times2}=\sqrt{25}\times\sqrt{2}\\&=5\times\sqrt{2}=5\sqrt{2}\end{aligned}$$

Thus we can express a pure surd as a mixed surd.

The laws of radicals enable us to express a pure surd as a mixed surd or a mixed surd as a pure surd.

Example 1. Express $\sqrt{98}$ as a mixed surd.

Solution.
$$\begin{aligned}\sqrt{98}&=\sqrt{7\times7\times2}\\&=\sqrt{7\times7}\times\sqrt{2}=7\times\sqrt{2}\\&=7\sqrt{2}\end{aligned}$$

Example 2. Express $\sqrt[3]{135}$ in simplest form.

Solution.
$$\begin{aligned}\sqrt[3]{135}&=\sqrt[3]{3\times3\times3\times5}\\&=\sqrt[3]{3^3}\times\sqrt[3]{5}.\\&=3\times\sqrt[3]{5}=3\sqrt[3]{5}\end{aligned}$$

Example 3. Express $7\sqrt{5}$ as a pure surd.

Solution.
$$\begin{aligned}7\sqrt{5}&=\sqrt{7\times7}\times\sqrt{5}\\&=\sqrt{49\times5}\qquad=\sqrt{245}\end{aligned}$$

Example 4. Reduce $4\sqrt[3]{5}$ to a pure surd.

Solution.
$$\begin{aligned}4\sqrt[3]{5}&=\sqrt[3]{4\times4\times4}\times\sqrt[3]{5}.\\&=\sqrt[3]{4\times4\times4\times5}\\&=\sqrt[3]{320}\end{aligned}$$

EXERCISE—3 *(a)*

(Section—A)

Simplify :

1. $\sqrt{63}$
2. $\sqrt{125}$
3. $\sqrt{486}$
4. $\sqrt[3]{250}$
5. $\sqrt[3]{648}$
6. $\sqrt[4]{243}$

Express as a pure surd :

7. $5\sqrt{7}$
8. $6\sqrt{10}$
9. $7\sqrt{11}$
10. $2\sqrt[3]{9}$
11. $3\sqrt[3]{12}$
12. $5\sqrt[4]{4}$

(Section—B)

13. Express $\sqrt[4]{567}$ as a mixed surd.
14. Express $\sqrt[6]{576}$ in simplest form.
15. Express $3\sqrt[4]{5}$ as a pure surd.
16. Reduce $3\sqrt[5]{4}$ to a pure surd.

3.4. SIMILAR SURDS

Two or more surds are said to be similar when they can be reduced as to have the same irrational factor.

For example, $3\sqrt{2}$, $5\sqrt{2}$, $8\sqrt{2}$ are similar surds. $\sqrt{3}$ and $4\sqrt{27}$ are also similar surds, for $4\sqrt{27}=4\times3\sqrt{3}$ *i.e.*, $12\sqrt{3}$.

Surds which are not similar are called dissimilar surds.

For example, $9\sqrt{5}$, $13\sqrt{2}$, $2\sqrt{7}$ are dissimilar surds.

3.5. COMPARISON OF SURDS

Two surds of the *same order* can be easily compared. We just compare their radicands. For example, $\sqrt[3]{20}>\sqrt[3]{12}$ because $20>12$

$$\sqrt[5]{31}<\sqrt[5]{53} \text{ because } 31<53.$$

If the surds are not of the same order, we first reduce them to the same order.

Example. Compare $\sqrt[4]{6}$, $\sqrt{2}$ and $\sqrt[3]{4}$.

Solution. The given surds are $\sqrt[4]{6}$, $\sqrt{2}$ and $\sqrt[3]{4}$. These surds are of the order 4, 2 and 3 respectively.

L.C.M. of 4, 2 and 3 is 12.

$$\sqrt[4]{6}=(6)^{1/4}=(6)^{3/12}=(6^3)^{1/12}=\sqrt[12]{216}$$

$$\sqrt{2}=(2)^{1/2}=(2)^{6/12}=(2^6)^{1/12}=\sqrt[12]{64}$$

$$\sqrt[3]{4}=(4)^{1/3}=(4)^{4/12}=(4^4)^{1/12}=\sqrt[12]{256}$$

Therefore $\sqrt[4]{256}>\sqrt[12]{216}>\sqrt[12]{64}$

i.e., $\sqrt[3]{4}>\sqrt[4]{6}>\sqrt{2}$

EXERCISE—3 (b)

(Section—A)

1. Reduce to surds of the same order :

(a) $\sqrt{3}, \sqrt[3]{4}, \sqrt[4]{5}$ (b) $\sqrt[3]{2}, \sqrt[6]{15}, \sqrt[3]{8}$

(c) $\sqrt[6]{4}, \sqrt[3]{3}, \sqrt[4]{2}$ (d) $\sqrt[4]{3}, \sqrt[8]{3}, \sqrt[6]{2}$.

2. Which of the following surds is greater ?

(a) $\sqrt[4]{5}$ or $\sqrt[3]{4}$ (b) $\sqrt[4]{10}$ or $\sqrt[3]{6}$

(c) $\sqrt[5]{2}$ or $\sqrt[8]{3}$ (d) $\sqrt[4]{4}$ or $\sqrt[5]{5}$.

(Section—B)

3. Arrange according to ascending order of magnitude :

(a) $\sqrt{3}, \sqrt[3]{5}, \sqrt[4]{6}$ (b) $\sqrt[6]{27}, \sqrt[3]{15}, \sqrt{21}$

(c) $\sqrt[3]{3}, \sqrt[4]{9}, \sqrt[6]{5}$ (d) $2\sqrt{5}, 3\sqrt{4}, \sqrt[4]{1000}$.

4. Arrange according to descending order of magnitude :

(a) $\sqrt[4]{3}, \sqrt[6]{10}, \sqrt[12]{25}$ (b) $\sqrt{3}, \sqrt[4]{10}, \sqrt[3]{6}$

(c) $\sqrt[3]{2}, \sqrt[6]{3}, \sqrt[9]{4}$ (d) $2\sqrt{3}, 3\sqrt{2}, \sqrt[5]{7}$.

3.6. SIMPLE AND COMPOUND SURDS

Surds consisting of one term only are called simple surds or monomial surds. For example, $\sqrt{7}$, $3\sqrt{5}$, $4\sqrt[3]{7}$, $-2\sqrt[5]{9}$ are simple surds.

An expression consisting of two or more simple surds connected by the sign + or − is called a compound surd.

For example, $\sqrt{2}+\sqrt{3}$, $\sqrt[3]{5}-\sqrt{2}+2\sqrt{3}$, are compound surds.

$\sqrt{5}-\sqrt{3}$ and $5\sqrt{2}+7\sqrt{5}$ are **binomial** surds.

$3+\sqrt{7}-\sqrt{2}$ and $\sqrt[3]{4}-2\sqrt{5}+\sqrt[4]{9}$ are **trinomial** surds.

3.7. ADDITION AND SUBTRACTION OF SURDS

Only *similar* surds can be added or subtracted. While adding or subtracting surds, we use distributive property to combine similar surds.

Thus,
$$7\sqrt{2}+3\sqrt{2}=(7+3)\sqrt{2}=10\sqrt{2}$$
$$8\sqrt{3}-2\sqrt{3}=(8-2)\sqrt{3}=6\sqrt{3}.$$

Example 1. Simplify : $\sqrt{12}-\sqrt{27}+\sqrt{75}$.

Solution. The given expression

$$\begin{aligned}
&=\sqrt{12}-\sqrt{27}+\sqrt{75}\\
&=\sqrt{4\times 3}-\sqrt{9\times 3}+\sqrt{25\times 3}\\
&=\sqrt{4}\times\sqrt{3}-\sqrt{9}\times\sqrt{3}+\sqrt{25}\times\sqrt{3}\\
&=2\sqrt{3}-3\sqrt{3}+5\sqrt{3}\\
&=(2+5)\sqrt{3}-3\sqrt{3}\\
&=7\sqrt{3}-3\sqrt{3}\\
&=(7-3)\sqrt{3}\\
&=4\sqrt{3}.
\end{aligned}$$

Example 2. Simplify : $2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}$.

Solution. The given expression

$$
\begin{aligned}
&=2\sqrt[3]{40}+3\sqrt[3]{625}-4\sqrt[3]{320}\\
&=2\sqrt[3]{8\times5}+3\sqrt[3]{125\times5}-4\sqrt[3]{64\times5}\\
&=2\sqrt[3]{8}\,\sqrt[3]{5}+3\sqrt[3]{125}\,\sqrt[3]{5}-4\sqrt[3]{64}\,\sqrt[3]{5}\\
&=2\times2\sqrt[3]{5}+3\times5\sqrt[3]{5}-4\times4\sqrt[3]{5}\\
&=4\sqrt[3]{5}+15\sqrt[3]{5}-16\sqrt[3]{5}\\
&=(4+15)\sqrt[3]{5}-16\sqrt[3]{5}\\
&=19\sqrt[3]{5}-16\sqrt[3]{5}\\
&=(19-16)\sqrt[3]{5}\\
&=3\sqrt[3]{5}.
\end{aligned}
$$

EXERCISE—3 (c)

(Section—A)

Simplify :

1. $3\sqrt{2}+2\sqrt{2}$.
2. $5\sqrt{3}-3\sqrt{3}$.
3. $\sqrt{50}+\sqrt{72}$.
4. $\sqrt{147}-\sqrt{75}$.
5. $\sqrt[3]{128}+\sqrt[3]{54}$.
6. $\sqrt[3]{375}-\sqrt[3]{192}$.

(Section—B)

Simplify :

7. $\sqrt{12}-\sqrt{27}+\sqrt{75}$.
8. $4\sqrt{5}+3\sqrt{20}-\sqrt{45}$.
9. $\sqrt{147}-\sqrt{108}+\sqrt{12}$.
10. $\sqrt{72}-2\sqrt{32}+\sqrt{8}$.
11. $4\sqrt[3]{27}+\sqrt[3]{8}+5\sqrt[3]{64}$.
12. $2\sqrt[3]{4}+5\sqrt[3]{32}-\sqrt[3]{108}$.

(Section—C)

Simplify :

13. $\sqrt{28}+\sqrt{63}-\sqrt{175}+2\sqrt{50}$.
14. $\sqrt{24}+\sqrt{5}-\sqrt{96}+\sqrt{125}$.
15. $5\sqrt{8}-\sqrt{54}+2\sqrt{27}-\sqrt{162}-\sqrt{108}$.
16. $\sqrt[4]{81}-8\sqrt[3]{216}+15\sqrt[5]{32}+\sqrt{225}$.

3.8. MULTIPLICATION OF SURDS

We have seen that $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$, where a and b are any two non-negative real numbers.

Surds of the *same* order can be multiplied to give a single surd of the same order, according to the following law :

$\sqrt[n]{a}\times\sqrt[n]{b}=\sqrt[n]{ab}$, where a and b are positive real numbers and n is a natural number.

Example 1. Multiply :

(*a*) $\sqrt[3]{7}$ by $\sqrt[3]{9}$ (*b*) $6\sqrt{5}$ by $13\sqrt{3}$.

Solution. (*a*) $\sqrt[3]{7}\times\sqrt[3]{9}=\sqrt[3]{7\times 9}$

$=\sqrt[3]{63}.$

(*b*) $6\sqrt{5}\times 13\sqrt{3}=(6\times 13)\,.\,\sqrt{5}\,\sqrt{3}$

$=78\sqrt{5\times 3}$

$=78\sqrt{15}.$

If the surds to be multiplied are of *different* orders, we reduce each surd to the same lowest order. Then we multiply them applying the above rule.

Example 2. Multiply : (*a*) $\sqrt[3]{5}$ by $\sqrt{6}$

(*b*) $5\sqrt{32}$ by $2\sqrt[3]{81}$.

Solution. (*a*) L.C.M. of 3 and 2 is 6.

So, both the surds should be of sixth order.

$$\sqrt[3]{5}=5^{1/3}=5^{2/6}=\sqrt[6]{5^2}=\sqrt[6]{25}$$

$$\sqrt{6}=6^{1/2}=6^{3/6}=\sqrt[6]{6^3}=\sqrt[6]{216}$$

$$\therefore \quad \sqrt[3]{5}\times\sqrt{6}=\sqrt[6]{25}\times\sqrt[6]{216}=\sqrt[6]{25\times 216}$$

$$=\sqrt[6]{5400}.$$

(*b*)

$$5\sqrt{32}\times 2\sqrt[3]{81}=5\sqrt{16\times 2}\times 2\sqrt[3]{27\times 3}$$

$$=[5\sqrt{16}\,\sqrt{2}]\times[2\sqrt[3]{27}\,\sqrt[3]{3}]$$

$$=[5\times 4\sqrt{2}]\times[2\times 3\sqrt[3]{3}]$$

$$=20\sqrt{2}\times 6\sqrt[3]{3}$$

$$=20\times 6\times\sqrt{2}\times\sqrt[3]{3}$$

$$=120\times\sqrt[6]{2^3}\times\sqrt[6]{3^2}$$

$$=120\times\sqrt[6]{8}\times\sqrt[6]{9}$$

$$=120\times\sqrt[6]{8\times 9}$$

$$=120\sqrt[6]{72}.$$

Note that *simplification* of surds is necessary before multiplication in the above example.

EXERCISE—3 (d)

(Section—A)

Simplify :

1. $\sqrt{3}\times\sqrt{7}$.
2. $\sqrt{6}\times\sqrt{11}$.
3. $\sqrt[3]{7}\times\sqrt[3]{9}$.
4. $\sqrt[4]{5}\times\sqrt[4]{8}$.
5. $5\sqrt{3}\times 3\sqrt{7}$.
6. $4\sqrt[5]{8}\times 5\sqrt[5]{9}$.
7. $\sqrt{2}\times\sqrt[3]{7}$.
8. $\sqrt[6]{8}\times\sqrt[4]{3}$.
9. $2\sqrt[3]{3}\times\sqrt{5}$.
10. $4\sqrt[3]{4}\times 5\sqrt[6]{10}$.

(Section—B)

Simplify and express the result in its simplest form :

11. $\sqrt[3]{18}\times\sqrt[3]{15}$.

12. $\sqrt{36}\times\sqrt{54}$.

13. $2\sqrt{20}\times\sqrt{45}$.

14. $3\sqrt{14}\times 2\sqrt{28}$.

15. $5\sqrt[3]{49}\times 2\sqrt[3]{14}$.

16. $2\sqrt[4]{45}\times 3\sqrt[4]{63}$.

17. $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[6]{5}$.

18. $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{4}$.

3.9. DIVISION OF SURDS

We have seen that $\sqrt{a}\div\sqrt{b}=\sqrt{\frac{a}{b}}$, where a and b are positive real numbers.

Surds of the *same* order can be divided to give a single surd of the same order, according to the following law :

$\sqrt[n]{a}\div\sqrt[n]{b}=\sqrt[n]{\frac{a}{b}}$, where a and b are positive real numbers and n is a natural number.

Example 1. Divide : (*a*) $\sqrt[3]{15}$ by $\sqrt[3]{3}$
(*b*) $3\sqrt{21}$ by $2\sqrt{35}$.

Solution. (*a*) $\sqrt[3]{15}\div\sqrt[3]{3}=\dfrac{\sqrt[3]{15}}{\sqrt[3]{3}}$

$$=\sqrt[3]{\frac{15}{3}}=\sqrt[3]{5}.$$

(*b*)
$$3\sqrt{21}\div 2\sqrt{35}=\frac{3\sqrt{21}}{2\sqrt{35}}$$

$$=\frac{3}{2}\sqrt{\frac{21}{35}}=\frac{3}{2}\sqrt{\frac{3}{5}}.$$

If the surds are of *different* orders, we reduce each surd to the same lowest order. Then we divide them applying the above law.

Example 2. Divide $5\sqrt[3]{4}$ by $3\sqrt{2}$.

Solution. $5\sqrt[3]{4}\div 3\sqrt{2}=\dfrac{5\sqrt[3]{4}}{3\sqrt{2}}$

$$=\frac{5\sqrt[6]{4^2}}{3\sqrt[6]{2^3}}=\frac{5\sqrt[6]{16}}{3\sqrt[6]{8}}$$

$$=\frac{5}{3}\sqrt[6]{\frac{16}{8}}=\frac{5}{3}\sqrt[6]{2}.$$

EXERCISE—3 (e)

(Section—A)

Simplify :

1. $\sqrt{12}\div\sqrt{6}$.
2. $\sqrt[3]{24}\div\sqrt[3]{6}$.
3. $\sqrt[4]{25}\div\sqrt[4]{15}$.
4. $\sqrt[5]{28}\div\sqrt[5]{8}$.
5. $3\sqrt{14}\div 5\sqrt{21}$.
6. $12\sqrt[3]{35}\div 4\sqrt[3]{50}$.

(Section—B)

7. $\sqrt[3]{5} \div \sqrt[4]{5}$.

8. $\sqrt{3} \div \sqrt[4]{5}$.

9. $\sqrt[4]{48} \div \sqrt[8]{72}$.

10. $5\sqrt[3]{6} \div 3\sqrt{10}$.

(Section—C)

11. Simplify : $\dfrac{\sqrt{75} \times \sqrt{60} \times \sqrt{63}}{\sqrt{40} \times \sqrt{200}}$.

12. Simplify : $\dfrac{2\sqrt{7} \times \sqrt{8} \times 5\sqrt{3}}{3\sqrt{3} \times \sqrt{15} \times \sqrt{42}}$.

The process of multiplication of compound surds is *similar* to that of compound algebraic expressions.

Example 1. Multiply $2\sqrt{12} - \sqrt{8}$ by $\sqrt{18}$.

Solution. $(2\sqrt{12} - \sqrt{8}) \times \sqrt{18}$

$$= 2\sqrt{12} \times \sqrt{18} - \sqrt{8} \times \sqrt{18}$$
$$= 2\sqrt{12 \times 18} - \sqrt{8 \times 18}$$
$$= 2\sqrt{6^2 \times 6} - \sqrt{12 \times 12}$$
$$= 2 \times 6\sqrt{6} - 12$$
$$= 12\sqrt{6} - 12.$$

Example 2. Multiply $3\sqrt{3} - 2\sqrt{2}$ by $\sqrt{2} - 2\sqrt{3}$.

Solution. $(3\sqrt{3} - 2\sqrt{2})(\sqrt{2} - 2\sqrt{3})$

$$= 3\sqrt{3}(\sqrt{2} - 2\sqrt{3}) - 2\sqrt{2}(\sqrt{2} - 2\sqrt{3})$$
$$= 3\sqrt{6} - 6 \times 3 - 2 \times 2 + 4\sqrt{6}$$
$$= 3\sqrt{6} - 18 - 4 + 4\sqrt{6}$$
$$= (3+4)\sqrt{6} - 22$$
$$= 7\sqrt{6} - 22.$$

EXERCISE—3 (f)

(Section—A)

Multiply :

1. $\sqrt{2} - \sqrt{6}$ by $\sqrt{2}$.

2. $\sqrt{6} + \sqrt{3}$ by $\sqrt{3}$.

3. $\sqrt{3} - 2\sqrt{5}$ by $\sqrt{3}$.

4. $\sqrt{10} - \sqrt{15}$ by $\sqrt{5}$.

5. $3\sqrt{5} - 2\sqrt{2}$ by $2\sqrt{5}$.

6. $2\sqrt{8} - \sqrt{18}$ by $3\sqrt{2}$.

(Section—B)

Simplify :

7. $(\sqrt{5} + \sqrt{6})(\sqrt{3} - \sqrt{2})$.

8. $(\sqrt{6} - \sqrt{3})(\sqrt{10} + \sqrt{5})$.

9. $(3\sqrt{18} + 2\sqrt{12})(\sqrt{50} - \sqrt{27})$.

10. $(\sqrt{5} + \sqrt{15})(2\sqrt{3} - 1)$.

11. $(4\sqrt{3} + 3\sqrt{2})(2\sqrt{5} - 5\sqrt{3})$.

12. $(3\sqrt{2} + \sqrt{5})(\sqrt{10} - 2\sqrt{2})$.

(Section—C)

13. Find the square of $(5\sqrt{3}-6\sqrt{2})$.

14. Evaluate $(2\sqrt{7}+3\sqrt{2})(2\sqrt{7}-3\sqrt{2})$.

15. Evaluate $(\sqrt{3}-\sqrt{2})^4$.

3.10. RATIONALISATION OF SURDS

When the product of two surds is a rational number, then each surd is called the rationalising factor of the other.

For example, $3\sqrt{2}\times\sqrt{2}=3\times\sqrt{2\times 2}=3\times 2$ *i.e.*, 6. Therefore $\sqrt{2}$ is the *rationalising factor* of $3\sqrt{2}$.

$$(\sqrt{5}+\sqrt{3})\times(\sqrt{5}-\sqrt{3})=(\sqrt{5})^2-(\sqrt{3})^2$$
$$=5-3 \text{ i.e., } 2$$

Therefore $(\sqrt{5}-\sqrt{3})$ is the *rationalising factor* of $(\sqrt{5}+\sqrt{3})$. Also $(\sqrt{5}+\sqrt{3})$ is the *rationalising factor* of $(\sqrt{5}-\sqrt{3})$.

The process of multiplying of a given surd by its rationalising factor to give a rational number as product is called **rationalisation** of a surd.

3.11. RATIONALISATION OF MONOMIAL SURDS

The rationalising factor of a monomial surd can be found easily, as illustrated below :

Example. Find the rationalising factor of

(*a*) $\sqrt[3]{25}$ (*b*) $3\sqrt[5]{4}\sqrt[4]{27}$

Solution. (*a*) Writing the surd in the index form, we have

$$\sqrt[3]{25}=\sqrt[3]{5^2} \quad \text{i.e.,} \quad 5^{2/3}$$

The fractional index $\frac{2}{3}$ will become 1 by adding $\frac{1}{3}$.

So, the rationalising factor is $5^{1/3}$ or $\sqrt[3]{5}$.

(*b*) Writing the given surd in the index form, we have

$$3\sqrt[5]{4}\sqrt[4]{27}=3.\sqrt[5]{2^2}\sqrt[4]{3^3} \quad \text{i.e.,} \quad 3.2^{2/5}.3^{3/4}$$

Now $\quad 2^{2/5}\times 2^{3/5}=2^{5/5} \quad \text{i.e.,} \quad 2$

and $\quad 3^{3/4}\times 3^{1/4}=3^{4/4} \quad \text{i.e.,} \quad 3$

Therefore the rationalising factor is $2^{3/5}.3^{1/4}$ *i.e.*, $\sqrt[5]{2^3}\sqrt[4]{3}$ or $\sqrt[5]{8}\sqrt[4]{3}$.

EXERCISE—3(g)

(Section—A)

Find the rationalising factor of :

1. $2\sqrt{3}$

2. $3\sqrt{2}$

3. $4\sqrt{5}$

4. $\sqrt{27}$

5. $\sqrt{45}$

6. $\sqrt{54}$

(Section—B)

Find the simplest rationalising factor of :

7. $\sqrt[3]{32}$

8. $\sqrt[3]{72}$

9. $\sqrt[3]{625}$

10. $\sqrt[3]{4}.\sqrt{3}$

11. $\sqrt{2}.\sqrt[3]{9}$

12. $\sqrt[3]{49}\,.\sqrt[4]{27}$

Sometimes surds occur in the denominator of an expression. It is almost always best to *rationalise* the denominator. So we multiply numerator and denominator of the expression by the *rationalising factor* of the denominator.

Example 1. Rationalise the denominator of

(*a*) $\frac{7}{\sqrt{5}}$ (*b*) $\frac{\sqrt{3}}{\sqrt{2}}$

Solution. (*a*) $\frac{7}{\sqrt{5}}=\frac{7\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}$ The rationalising factor of √5 is √5.

$$=\frac{7\sqrt{5}}{5}.$$

(*b*) $\frac{8\sqrt{3}}{\sqrt{2}}=\frac{8\sqrt{3}\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$ The rationalising factor of √2 is √2.

$$=\frac{8\sqrt{3\times2}}{2}$$

$$=4\sqrt{6}.$$

Example 2. Simplify : $7\sqrt{3}\div4\sqrt{2}$.

Solution. $7\sqrt{3}\div4\sqrt{2}=\frac{7\sqrt{3}}{4\sqrt{2}}$

Rationalising the denominator, we get

$$\frac{7\sqrt{3}}{4\sqrt{2}}=\frac{7\sqrt{3}\times\sqrt{2}}{4\sqrt{2}\times\sqrt{2}}=\frac{7\sqrt{3\times2}}{4\sqrt{2\times2}}$$

$$=\frac{7\sqrt{6}}{4\times2}\quad=\frac{7\sqrt{6}}{8}.$$

Example 3. Given that $\sqrt{5}=2{\cdot}236$ approx., find the value of $\frac{3}{2\sqrt{5}}$ upto 3 places of decimals.

Solution. $\frac{3}{2\sqrt{5}}=\frac{3\times\sqrt{5}}{2\sqrt{5}\times\sqrt{5}}$

$$=\frac{3\sqrt{5}}{2\times5}\quad=\frac{3\sqrt{5}}{10}$$

$$=\frac{3\times2{\cdot}236}{10}$$

$$=\frac{6{\cdot}708}{10}$$

$=0{\cdot}6708$ *i.e.*, $0{\cdot}671$.

EXERCISE—3 h)

(Section—A)

Express the following with rational denominators :

1. $\frac{6}{\sqrt{3}}$
2. $\frac{2}{\sqrt{7}}$
3. $\frac{3\sqrt{2}}{\sqrt{6}}$
4. $\frac{4}{5\sqrt{3}}$
5. $\frac{6}{5\sqrt{3}}$
6. $\frac{4\sqrt{6}}{3\sqrt{2}}$

(Section—B)

Simplify :

7. $\sqrt{\frac{98}{3}}$ 8. $5\sqrt{3} \div 4\sqrt{2}$ 9. $\frac{3\sqrt[3]{5}}{\sqrt[3]{9}}$

(Section—C)

10. Given that $\sqrt{5}=2{\cdot}236$ approx. find to three places of decimals, the value of $\frac{2}{\sqrt{5}}$.

11 Given $\sqrt{2}=1{\cdot}414$, find the value of $\frac{3}{2\sqrt{2}}$ upto 3 places of decimals.

12. Given $\sqrt{3}=1{\cdot}732$, find the value of $\frac{8}{2\sqrt{3}}$ upto 3 places of decimals.

3.12. RATIONALISATION OF BINOMIAL QUADRATIC SURDS

Binomial quadratic surds are said to be *conjugate* to each other when they differ only in the sign connecting their terms.

For example, $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$ are *conjugate* to each other.

Similarly $2\sqrt{3}-\sqrt{2}$ and $2\sqrt{3}+\sqrt{2}$ are *conjugate* to each other.

$$(\sqrt{7}-\sqrt{3})\times(\sqrt{7}+\sqrt{3})=(\sqrt{7})^2-(\sqrt{3})^2$$

$$=7-3 \qquad =4 \text{ which is a rational number.}$$

Therefore $(\sqrt{7}+\sqrt{3})$ is the *rationalising factor* of $(\sqrt{7}-\sqrt{3})$. Also $(\sqrt{7}-\sqrt{3})$ is the *rationalising factor* of $(\sqrt{7}+\sqrt{3})$. But $(\sqrt{7}+\sqrt{3})$ and $(\sqrt{7}-\sqrt{3})$ are *conjugate* to each other.

Hence, *the simplest rationalising factor of a binomial quadratic surd is its conjugate surd.*

Example 1. Express $\frac{3}{5-\sqrt{2}}$ with a rational denominator.

Solution. The rationalising factor of $(5-\sqrt{2})$ is $(5+\sqrt{2})$.

$$\frac{3}{5-\sqrt{2}}=\frac{3}{5-\sqrt{2}}\times\frac{5+\sqrt{2}}{5+\sqrt{2}}$$

$$=\frac{3(5+\sqrt{2})}{(5-\sqrt{2})(5+\sqrt{2})}$$

$$=\frac{3(5+\sqrt{2})}{25-2} \qquad =\frac{15+3\sqrt{2}}{23}$$

Example 2. Rationalise the denominator of $\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{6}+5}$

Solution. The rationalising factor of $5+2\sqrt{6}$ is $5-2\sqrt{6}$.

$$\frac{2\sqrt{3}-3\sqrt{2}}{2\sqrt{6}+5}=\frac{2\sqrt{3}-3\sqrt{2}}{5+2\sqrt{6}}\times\frac{5-2\sqrt{6}}{5-2\sqrt{6}}$$

$$=\frac{10\sqrt{3}-15\sqrt{2}-4\sqrt{18}+6\sqrt{12}}{(5)^2-(2\sqrt{6})^2}$$

$$=\frac{10\sqrt{3}-15\sqrt{2}-4.3\sqrt{2}+6.2\sqrt{3}}{25-4\times 6}$$

$$=\frac{10\sqrt{3}-15\sqrt{2}-12\sqrt{2}+12\sqrt{3}}{25-24}$$

$$=\frac{(10+12)\sqrt{3}-(15+12)\sqrt{2}}{1}$$

$$=22\sqrt{3}-27\sqrt{2}.$$

Example 3. Express $\dfrac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{10}-\sqrt{5}}$ with a rational denominator.

Solution. Here the denominator is a *trinomial.* We write the trinomial as a binomial by grouping together two of the terms.

$\sqrt{3}+\sqrt{10}-\sqrt{5}$ can be rewritten as $(\sqrt{3}+\sqrt{10})-\sqrt{5}$

Therefore, the *rationalising factor of* $(\sqrt{3}+\sqrt{10})-\sqrt{5}$ is $(\sqrt{3}+\sqrt{10})+\sqrt{5}$

$$\frac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{10}-\sqrt{5}}=\frac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{(\sqrt{3}+\sqrt{10})-\sqrt{5}}$$

$$=\frac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{(\sqrt{3}+\sqrt{10})-\sqrt{5}}\times\frac{(\sqrt{3}+\sqrt{10})+\sqrt{5}}{(\sqrt{3}+\sqrt{10})+\sqrt{5}}$$

$$=\frac{[(\sqrt{10}+\sqrt{5})-\sqrt{3}]\times[(\sqrt{10}+\sqrt{5})+\sqrt{3}]}{(\sqrt{3}+\sqrt{10})^2-(\sqrt{5})^2}$$

$$=\frac{(\sqrt{10}+\sqrt{5})^2-(\sqrt{3})^2}{3+10+2\sqrt{3}\sqrt{10}-5}$$

$$=\frac{10+5+2\sqrt{10}\sqrt{5}-3}{8+2\sqrt{30}}$$

$$=\frac{12+2\sqrt{50}}{8+2\sqrt{30}}\qquad=\frac{2(6+\sqrt{50})}{2(4+\sqrt{30})}$$

$$=\frac{6+5\sqrt{2}}{4+\sqrt{30}}\times\frac{4-\sqrt{30}}{4-\sqrt{30}}$$

$$=\frac{24+20\sqrt{2}-6\sqrt{30}-5\sqrt{60}}{(4+\sqrt{30})(4-\sqrt{30})}$$

$$=\frac{24+20\sqrt{2}-6\sqrt{30}-5\times 2\sqrt{15}}{16-30}$$

$$=\frac{24+20\sqrt{2}-6\sqrt{30}-10\sqrt{15}}{-14}$$

$$=\frac{-(6\sqrt{30}+10\sqrt{15}-20\sqrt{2}-24)}{-14}$$

$$=\frac{-2(3\sqrt{30}+5\sqrt{15}-10\sqrt{2}-12)}{-14}$$

$$=\frac{3\sqrt{30}+5\sqrt{15}-10\sqrt{2}-12}{7}$$

EXERCISE—3(i)

(Section—A)

Rationalise the denominator of the following :

1. $\dfrac{1}{\sqrt{3}+\sqrt{2}}$
2. $\dfrac{4}{\sqrt{5}-\sqrt{3}}$
3. $\dfrac{1}{4+\sqrt{5}}$
4. $\dfrac{6}{\sqrt{3}-\sqrt{6}}$
5. $\dfrac{3\sqrt{7}}{5-\sqrt{7}}$
6. $\dfrac{3\sqrt{2}}{5+\sqrt{2}}$

(Section—B)

Express with rational denominator :

7. $\dfrac{3+\sqrt{2}}{3-\sqrt{2}}$
8. $\dfrac{5+\sqrt{2}}{5-\sqrt{2}}$
9. $\dfrac{6}{3\sqrt{2}-2\sqrt{3}}$
10. $\dfrac{2+3\sqrt{2}}{3-2\sqrt{2}}$
11. $\dfrac{4\sqrt{2}+\sqrt{5}}{3\sqrt{3}-2\sqrt{3}}$
12. $\dfrac{1+3\sqrt{2}}{2\sqrt{5}+3}$
13. Given $\sqrt{3}=1{\cdot}732$, find the value of $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.
14. Evaluate $x^2+\dfrac{1}{x^2}$, when $x=5+2\sqrt{6}$.
15. If $x=3-\sqrt{5}$, find the value of $x^2+\dfrac{16}{x^2}$.

(Section—C)

16. Given $\sqrt{2}=1{\cdot}4142$ and $\sqrt{3}=1{\cdot}7321$, find correct to three places of decimals, the value of

$$\frac{4}{3\sqrt{3}-2\sqrt{2}}+\frac{3}{3\sqrt{3}+2\sqrt{2}}$$

Simplify each of the following by rationalise the denominator :

17. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$
18. $\dfrac{12}{3+\sqrt{5}-2\sqrt{2}}$

REVIEW EXERCISE—III

(Section—A)

1. Fill in the blanks to make a true statement in each case :

   (*a*) $\sqrt[5]{10}$ is a surd of .... order.

   (*b*) A mixed can be expressed as a .... surd.

   (*c*) A compound surd has at least .... terms.

   (*d*) We can add or subtract only .... surds.

   (*e*) The conjugate surd of $2\sqrt{3}-\sqrt{5}$ is ....

   (*f*) The rationalising factor of $4\sqrt{5}$ is ....

   (*g*) The rationalising factor of a binomial quadratic surd is its . .. . surd.

   (*h*) A complete surd can be expressed as a .... surd.

2. Simplify : $5\sqrt{3}+2\sqrt{27}+\dfrac{1}{\sqrt{3}}$

3. Simplify : $\sqrt{252}-5\sqrt{6}+\sqrt{294}-3\sqrt{\dfrac{1}{6}}$

4. Divide $\sqrt{24}$ by $\sqrt[3]{200}$.

(Section—B)

5. Prove that $\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}$ is rational.

6. Express $\dfrac{2\sqrt{5}}{\sqrt{5}-\sqrt{2}}$ with a rational denominator.

7. Simplify : $\dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{80}+\sqrt{48}-\sqrt{45}-\sqrt{27}}$.

8. Find the value of $\dfrac{6}{2\sqrt{3}-\sqrt{6}}+\dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\dfrac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

(Section—C)

9. Express $\dfrac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}}$ with a rational denominator.

10. If $x=3+\sqrt{8}$, find the value of $x^4+\dfrac{1}{x^4}$.

11. Find the value of $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$ correct to three decimal places.

12. If $x=3-\sqrt{8}$, find the value of $x^6+\dfrac{1}{x^6}$.

□□

4

FUNCTIONS

4.1. GRAPHS

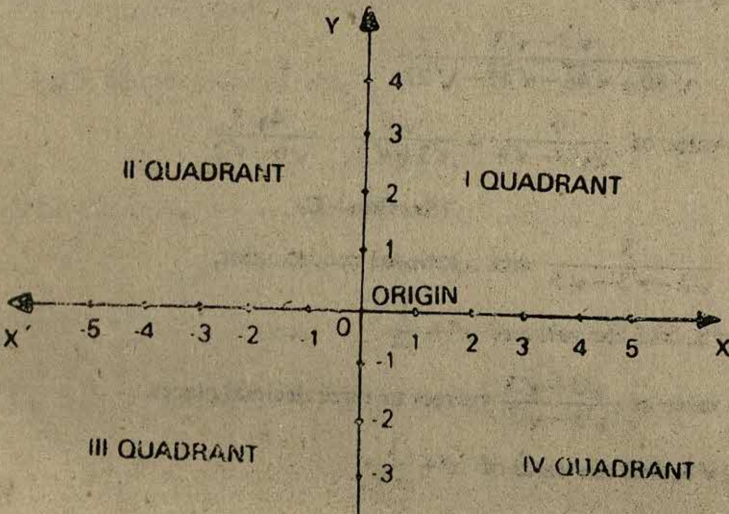
Graphs are generally used in various fields of life. A graph is a pictorial representation. Its object is to convey quickly as much information as possible about variations in two related quantities. It is necessary to know how to fix the positions of points which form the graph.

4.2. RECTANGULAR COORDINATE SYSTEM

Let us draw any line $X'OX$ and through O let us draw another line $Y'OY$ perpendicular to $X'OX$.

Since there is a one-one correspondence between the real numbers and points on a line, real numbers may be represented on either of these two lines.

Let us take the point of intersection O of these two lines as the point corresponding to the real number 0 on either line.



We know that every point of a line represents a real number, and conversely, every real number is represented by a unique point on the line.

So we represent all positive real numbers on $X'OX$ along OX and negative real numbers along OX'.

Similarly we represent all positive real numbers on $Y'OY$ along OY and negative real numbers along OY'.

Lines $X'OX$ and YOY' are called **co-ordinate axes** or **axes of reference.**

The horizontal line $X'OX$ is called **axis of x** or **x-axis.**

The vertical line YOY' is called **axis of y** or **y-axis.**

The point O is called the **origin.**

The axis of co-ordinates divide the plane into four regions which are called **quadrants** XOY, YOX', $X'OY'$ and $Y'OX$ are respectively called the first, second, third and fourth quadrants.

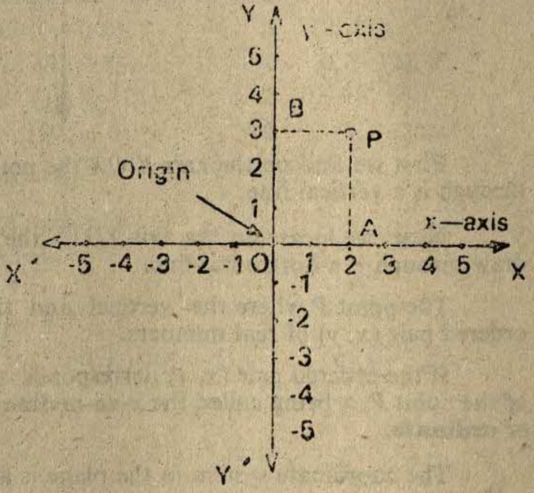
A point P in the plane can be assigned an ordered pair of numbers called co-ordinates, as shown in the figure.

(*i*) BP is called the X-co-ordinate or **abscissa** of P.

(*ii*) AP is called the Y-co-ordinate or **ordinate** of P.

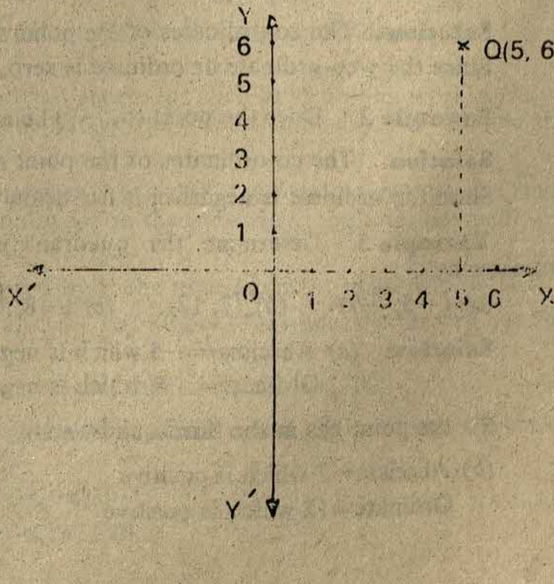
(*iii*) We always write the abscissa first. Thus the co-ordinates of P are (2, 3).

The co-ordinates of the origin are (0, 0).

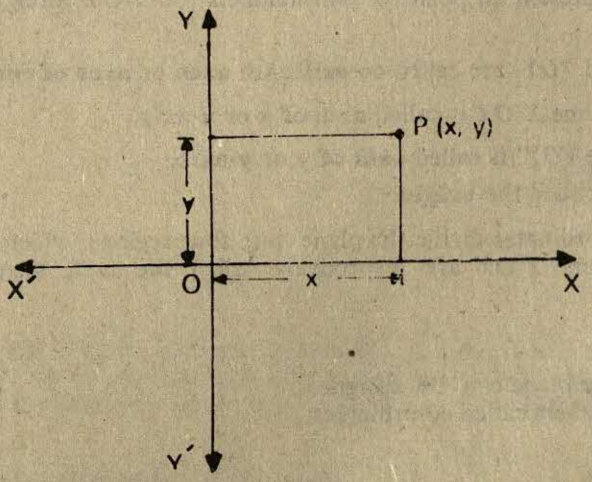


Conversely an ordered pair of numbers (5, 6) for example, can be assigned to a point in the plane as follows. We measure 5 units along the X-axis and then 6 units along a line parallel to the Y-axis arriving at point Q. You will note that the ordered pair (6, 5) will denote a different point.

Hence we say that there is a one-to-one correspondence between the points in the plane and the ordered pairs of real numbers.



If we are given an ordered pair (x, y) of real numbers, we can represent it by a point on the plane as follows :



First we find on the axis $X'OX$ the point corresponding to the real number x and draw through it a vertical line.

Next, we locate on the axis YOY' the point corresponding to the real number y and draw through it a horizontal line.

The point P where the vertical and the horizontal lines meet is said to represent the ordered pair (x, y) of real numbers.

If the ordered pair (x, y) corresponds to the point P, we say (x, y) are the co-ordinates of the point P, x being called the **x-co-ordinate** or **abscissa,** y being called the **y-co-ordinate** or **ordinate.**

The coordinate system in the plane is also known as the rectangular co-ordinate system.

Example 1. If a point has co-ordinates $(-4, 0)$, does it lie on the x-axis or y-axis ?

Solution. The co-ordinates of the point are $(-4, 0)$.
Since the y-co-ordinate or ordinate is zero, it lies on the x-axis.

Example 2. Does the point $(5, -7)$ lie above the x-axis or below the x-axis ?

Solution. The co-ordinates of the point are $(5, -7)$.
Since the ordinate is negative, it lies below the x-axis.

Example 3. Determine the quadrant in which the points with the following co-ordinates lie :

(*a*) $(-5, -9)$ (*b*) $(7, 12)$ (*c*) $(-8, 10)$ (*d*) $(9, -16)$

Solution. (*a*) Abscissa$=-5$ which is negative
Ordinate$=-9$ which is negative

So the point lies in the third quadrant.

(*b*) Abscissa$=7$ which is positive
Ordinate$=12$ which is positive

So the point lies in the first quadrant.

(*c*) Abscissa $= -8$ which is negative
Ordinate $= 10$ which is positive.
So the point lies in the second quadrant.

(*d*) Abscissa $= 9$ which is positive
Ordinate $= -16$ which is negative
So the point lies in the fourth quadrant.

EXERCISE—4 (a)

(Section—A)

Plot on a graph paper the point whose coordinates are given below :

1. $(5, 9)$. **2.** $(-5, 1)$. **3.** $(-7, 0)$. **4.** $(0, 11)$.
5. $(-6, -13)$ **6.** $(0, -15)$. **7.** $(9, -16)$. **8.** $(0, -12)$.
9. $(7, 0)$. **10.** $(-16, -16)$. **11.** $(10, -10)$. **12.** $(-18, 18)$.
13. $(4, 20)$. **14.** $(-8, 22)$. **15.** $(22, -6)$. **16.** $(-9, -24)$.

(Section—B)

17. Name the quadrants in which the following points lie :
(*a*) (x, y) where x is negative and y is positive.
(*b*) (p, q) where p and q are both negative ?
(*c*) (r, s) where r is negative and s is positive ?
(*d*) $(-7, 5)$, (*e*) $(-5, -5)$, (*f*) $(0, -5)$, (*g*) $(-3, 0)$.
(*h*) $(0, 0)$, (*i*) $(3, 0)$.

4.3. FUNCTIONS

Function is an important concept in mathematics. We will explain the basic idea by a simple example.

Suppose that 16 boys were given work in a factory on a particular day. Each boy is paid Rs. 10 as daily wages

Can you tell the total payment made by the factory for such boys on any working day ? If we know the number of boys employed by the factory on a particular day, we can find the total payment made.

If 20 boys were employed on first Monday of April '88, the total payment to be made to them will be Rs. 200. If 25 boys were employed on the next day, the total payment will be Rs. 250 and so on.

If we denote the number of boys employed by x, the total payment made will be $10x$.

x can take integral values between 0 and 30 *i.e.*, $0 \leqslant x \leqslant 30$ since the factory has the capacity to employ 30 boys.

Then $10x$ will take integral values between 0 and 300 *i.e.*, $0 \leqslant 10x \leqslant 300$.

Let us denote the set of non-negative integers between 0 and 30 by X and the set of non-negative integers between 0 and 300 by Y.

Then $X = \{x : x \in I \text{ and } 0 \leqslant x \leqslant 30\}$
$Y = \{y : y \in I \text{ and } 0 \leqslant y \leqslant 300\}$

Thus with every element $x \in X$, we associate an element $y \in Y$ such that $y=10x$. This relationship is described by saying that y is a *function* of x, where $y=10x$.

Note that each element of X is associated with one element of Y. But every element of Y is *not* necessarily associated with some element of X. $25 \in Y$, but 25 is *not* associated with any element of X. Only multiples of 10 are associated with some elements of x because $y=10x$.

Also note that the same element of Y may be associated to several elements of X. But each element of X will be associated with *exactly one element* of Y.

Let X and Y be any two non-empty sets. If every element $x \in X$ is associated by a rule, with one and only one element of Y, then this association determines a function from X to Y.

The set X is called the **domain** of the function and the set Y is called the **co-domain** or **range** of the function.

Functions are usually denoted by single letters such as F, f, G, g.

A variable such as x, is used to represent the elements of domain. The corresponding elements of range are then denoted by $f(x)$ or $F(x)$.

We read the symbol $f(x)$ as "*f at x*" or "*f of x*".

$f(x)$ is called the **image** of x and x is called **pre-image** of $f(x)$.

The ordered pairs of a function f are represented as $[x, f(x).]$

Sometimes, in symbols, we write like this :

$$f=\{(x, f(x)) : f(x)=x+2\}$$

OR

$$f=\{(x, y) : y=x+2\}.$$

Considering that a function f from A to B is also the arrow diagram, we say f maps A into B and we write symbolically as under :

(*i*) $f : A \longrightarrow B$. Read as '*f maps A into B*'.

(*ii*) $A \xrightarrow{f} B$. Read as '*A is mapped into B* by *f*'.

(*iii*) $A \longrightarrow B,\ x \longrightarrow x+2$. Read as '*function f maps A into B such that each element x of A is mapped into* $x+2$.

A function is also known as **mapping.**

A function is said to be a real function if its domain and range are subsets of real numbers.

Here we will study only real functions. Let $y=f(x)$ be a real function.

x can assume any value in the domain of f. Then y will have a definite value corresponding to that value of x. Thus the value of y will depend on the value of x. That is why x is called **independent variable** and y is called **dependent variable**.

Consider a real function given by

$$y=f(x)=3x^2-1$$

When $x=1$, $\quad y=f(1)=3\times 1^2-1 \quad =2$

When $x=-2$, $\quad y=f(-2)=3\times(-2)^2-1=11$

$f(1)$ is the **value of the function** at $x=1$.

Then $f(1)=2$. Similarly $f(-2)$ is the **value** of the function at $x=-2$. Then $f(-2)=11$,

Example 1. Let $f : A \to N$, $x \to x^2$, where
$A=\{1, 2, 3, 4, 5\}$. Express f in the roster form.
Find the domain and range of the function.

Solution. Given $A=\{1, 2, 3, 4, 5\}$
$f : x \to x^2$

Domain of f is $\{1, 2, 3, 4, 5\}$
Then $f : 1 \to 1, 2 \to 4, 3 \to 9, 4 \to 16, 5 \to 25$.
Hence $f=\{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\}$
Range of f is $\{1, 4, 9, 16, 25\}$.

Example 2. If the real function f is defined such that its value at x is

$$f(x)=\sqrt{(x-4)(7-x)}, \text{ find its domain.}$$

Solution. Given
$$f(x)=\sqrt{(x-4)(7-x)}$$
$$=\sqrt{x-4}\ \sqrt{7-x}$$

We must find x when $\sqrt{x-4}$ and $\sqrt{7-x}$ are defined.

For $\sqrt{x-4}$ to be well defined, we must have $x-4 \geqslant 0$ *i.e.*, $x \geqslant 4$.

Also for $\sqrt{7-x}$ to be well defined, we must have

$$7-x \geqslant 0 \text{ or } 7 \geqslant x \text{ i.e., } x \leqslant 7.$$

Combing the two, we find that the function is defined when $4 \leqslant x \leqslant 7$.

Hence the domain of the given function is the set of real numbers between 4 and 7. We can write it as

Domain of $f=\{x : x \text{ is real and } 4 \leqslant x \leqslant 7\}$

EXERCISE—1 (b)

(Section—A)

1. If a real function f is defined such that its value at x is given by $f(x)=\sqrt{x}$, find the domain of f.
2. Let $f :\ N \to R$ be given by
$$f(x)=+\sqrt{x+3}-3$$
Find the value of (*i*) $f(1)$ (*ii*) $f(13)$ [*C.B.S E., 1980* [*A.I.*)]
3. $A=\{1, 2, 3, 4, 5, 6\}$, $B=\{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and
$S=\{(x, y), : y=2x-5, x \in A, y \in B\}$.
   - (*i*) List the elements of S.
   - (*ii*) List the domain of S.
   - (*iii*) List the range of S.
   - (*iv*) What is the domain of the variable x ?
4. Write down the domain and range of the function.
$\{(x, y) : x=3y$ and x and y are natural numbers less than $10\}$.

5. If $f : A \rightarrow N, x \rightarrow 2x^2 - 1, x \in A$, where
$$A = \{1, 2, 3, 4, 5\}.$$
Write the function in roster form.
Give its domain and range.

(Section—B)

6. Write down the functions from R to R defined below in the roster form. Draw their graphs.
(*a*) $f : x \rightarrow 1/x, x \in \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}\}$
(*b*) $f : x \rightarrow 2x, x \in N$.

7. If $g = \{(x, y) : y = x^2 - 1\}$ find
(*a*) $g(0)$ (*b*) $g(-1)$ (*c*) $g(1)$ (*d*) $g(-3)$.

8. If $f = \{(x, f(x) : f(x) = x - 2\}$, find
(*a*) $f(2)$, (*b*) $f(8)$ (*c*) $f(0)$, (*d*) $f(\sqrt{2})$.

9. Let $f(x) = \dfrac{2x^2 - 1}{x - 2}, x \neq 2, x \in R$,
find the value of $\dfrac{f(5)}{f(\frac{1}{2})}$.

10. Given that $f(x) = 2x$ and $f : N \rightarrow N$
(*a*) Find the range of f.
(*b*) Find the domain of f.
(*c*) Does $f(2) + f(3) = f(2+3)$?
(*d*) Does $f(2) . f(3) = f(2 . 3)$?
(*e*) Does $f(5x) = 5 f(x)$?

(Section—C)

11. Let $f : R \rightarrow R$ be given by $f(x) = x^2 + 2x + 1$.
Find the value of (*i*) $f(-1) + f(1)$ (*ii*) $f(2) \times f(3)$
Is $f(-1) + f(1) = f(0)$?
Is $f(2) \times f(3) = f(6)$?

12. Let 'f' be defined by $f(x) = \dfrac{x}{x^2 + 1}, x \in R$.
Find (*i*) $f\left(\dfrac{1}{x}\right), x \neq 0$ (*ii*) $f(2x)$
(*iii*) $f(x - 1)$ (*iv*) $f\{f(-1)\}$

13. If $f(x) = \dfrac{ax - b}{cx - a} = y$, show that $f(y) = x$.

4.4. GRAPH OF FUNCTIONS

Let us now learn how to sketch the graphs of real functions.
Let f be a function, defined as $f(x) = 2x + 5$.

The value $2x + 5$ at x is defined for all real values of x. Therefore, the domain of the function is the set of all real numbers.

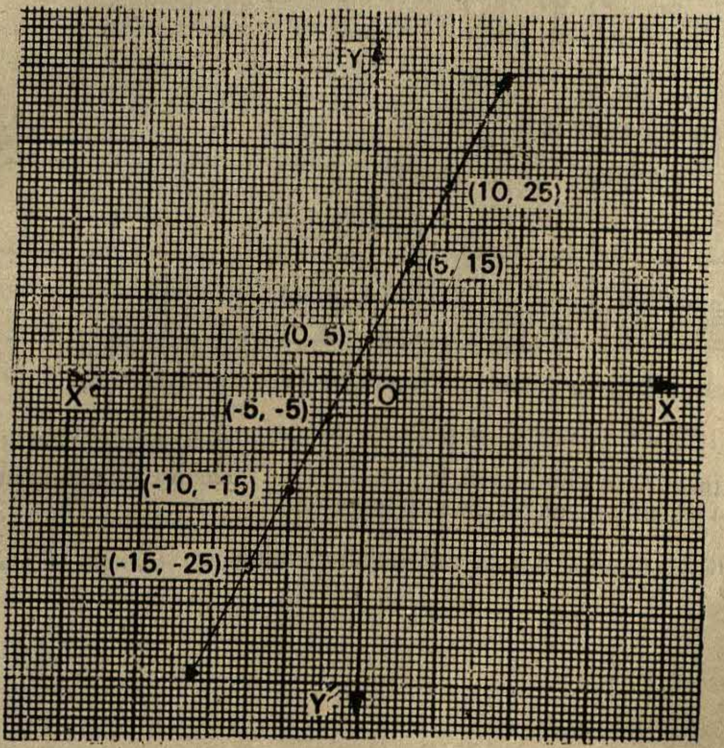
We now choose various values of x in the domain and calculate the values of $f(x)$ as under :

When $x=0$, $f(x)=2x+5=2\times 0+5 \quad =5$
When $x=5$, $f(x)=2x+5=2\times 5+5 \quad =15$
When $x=10$, $f(x)=2x+5=2\times 10+5 \quad =25$
When $x=-5$, $f(x)=2x+5=2\times(-5)+5 =-5$
When $x=-10$, $f(x)=2x+5=2\times(-10)+5=-15$
When $x=-15$, $f(x)=2x+5=2\times(-15)+5=-25$

We tabulate these as follows :

x	0	5	10	-5	-10	-15
$f(x)$	5	15	25	-5	-15	-25

We take the points, in a rectangular coordinate system, with abscissas as different values of x in the table and ordinates the corresponding values of $f(x)$. Therefore we plot the points (0, 5), (5, 15), (10, 25), $(-5, -5)$, $(-10, -15)$, $(-15, -25)$. Then we join all these points.



The line so obtained is the **graph** of the given function, $f(x)=2x+5$.

Functions of the form $\mathbf{f(x)=ax+b}$, where a and b are real numbers, are called **linear functions.**

The graph of a linear function is always a line.

We should assign only *integral values* to x to find the values of $f(x)$. This will give only integers for abscissas and ordinates. Plotting of such coordinate is easier

Although we need to plot *only two points* to determine the graph of a linear function over real numbers, it is a good practice to plot several points as a check.

Let f be a function defined as $f(x)=x^2-3x-4$.

We shall now sketch the graph of f.

Here x^2-3x-4 is defined for all real values of x. Therefore, the domain of f is the set of all real numbers.

We assign various real values to x and find the corresponding values of $f(x)$.

When $x=0$, $f(0)=0-3\times0-4=-4$

$x=2$, $f(2)=4-6-4=-6$

$x=4$, $f(4)=16-12-4=0$

$x=5$, $f(5)=25-15-4=6$

$x=6$, $f(6)=36-18-4=14$

$x=-1$, $f(-1)=1+3-4=0$

$x=-2$, $f(-2)=4+6-4=6$

$x=-3$, $f(-3)=9+9-4=14$

We tabulate these values as under.

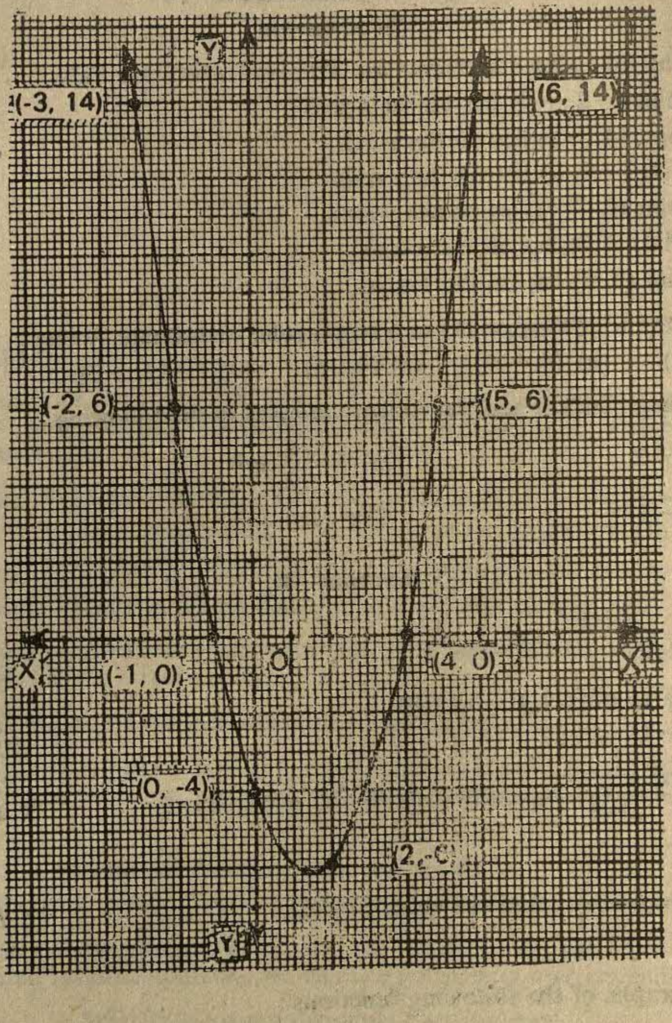
x	0	2	4	5	6	-1	-2	-3
$f(x)$	-4	-6	0	6	14	0	6	14

We plot the following points in a rectangular coordinate system.

$(0, -4)$, $(2, -6)$, $(4, 0)$, $(5, 6)$, $(6, 14)$, $(-1, 0)$, $(-2, 6)$, $(-3, 14)$.

We join these points by smooth free hand curve. Thus, we get the **graph** of the function $f(x)=x^2-3x-4$ which is given on the next page, 65.

Note that this graph is a *curve* different from a line.



The graph of a quadratic function is a curve called **parabola.** *The curve is concave upwards and is symmetric to y-axis.*

EXERCISE—4 (c)

Sketch the graph of the following functions :

(Section—A)

1. $f(x)=x+2$
2. $f(x)=3-x$
3. $f(x)=5-3x$
4. $f(x)=7+4x$
5. $f(x)=\dfrac{2x+3}{5}$
6. $f(x)=\dfrac{12-x}{6}$.

(Section—B)

7. $f(x)=x^2+2$
8. $f(x)=x^2-3$
9. $f(x)=2-x^2$
10. $f(x)=(x-2)^2$

REVIEW EXERCISE—IV

(Section—A)

1. Which of the following statements are True and which are False ?
   (*a*) The point $(-2, -2)$ lies on the x-axis.
   (*b*) The point $(4, -5)$ lies on the y-axis.
   (*c*) The point $(-9, 0)$ lies on the y-axis.
   (*d*) The point $(0, -6)$ lies on the y-axis.
   (*e*) The point $(7, 0)$ lies on the x-axis.
   (*f*) The point $(0, 8)$ lies on the x-axis.

2. Represent the following points in a rectangular coordinate system :
   (*a*) $(-7, 0)$ (*b*) $(4, -9)$ (*c*) $(9, 15)$ (*d*) $(-12, -7)$

3. Name the independent and dependent variables in the following :
   (*a*) $3y=x^2-2$ (*b*) $V=\frac{4}{3}\pi r^3$
   (*c*) $y=\sqrt{x-1}$ (*d*) $y=3x+5$

(Section—B)

4. Let $f(x)=\frac{x+3}{2x-1}$, $x\neq\frac{1}{2}$, $x\in R$.

   Find the value of $f(1), f(-1), f(0)$.
   Is $f(1)+f(-1)=f(0)$? [*C.B.S.E., 1983 (A.I.)*]

5. If $f(x)=\frac{2x^2+3x+1}{x+1}$, verify if $f(1)+f(2)=f(3)$ is true. [*C.B.S.E., 1983 (Delhi)*]

6. Given $f(x)=\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$, find the value of $f(2)$. [*C.B.S.E., 1979 (Delhi)*]

Draw the graphs of the following functions :

7. $f(x)=3-\frac{x}{2}$
8. $f(x)=\frac{x-7}{2}$
9. Let $f(x)=\frac{x+3}{2x-1}$, $x\in R$, $x\neq\frac{1}{2}$.

   Find (*i*) $f(0)$ (*ii*) $f(-3)$ (*iii*) value of x such that $f(x)=4$
   [*C.B.S.E., 1980 (Delhi)*]

(Section—C)

10. Let f be a function defined as $f(x)=x^2$. Sketch the graph of f.
11. Sketch the graph of a function defined as $f(x)=x^2+x-6$.

□ □

5

POLYNOMIALS

5.1. POLYNOMIALS

We are familiar with the concept of function. Here we shall discuss a special type of function, known as **polynomial**. Polynomials are a special type of algebraic expressions involving only one variable.

Let us consider $p(x)=5x^2-3x+2$. This is a function of x.

Then $p(1)=5-3+2=4$; $p(2)=20-6+2=16$

$p(-1)=5+3+2=10$; $p(0)=0-0+2=2$.

Therefore $p(x)$ is a function whose domain is the set all real numbers.

Note that here powers of x in each term are only *non-negative integers* and the coefficient of each power of x is a *real number*.

Such functions are called **polynomial over real numbers**.

The function $f(x)=a_0+a_1x+a_2x^2+...+a_nx^n$, where x is any real number, n is any non-negative integer and $a_1, a_2, a_3, ..., a_n$ are given real numbers and $a_n \neq 0$, is called a **polynomial** in x over reals.

The domain of a polynomial function is the real set R.

The real numbers $a_0, a_1, a_2, ..., a_n$ are called the **coefficients** of the polynomials.

If the coefficients of the powers of x are all integers, we call it a *polynomial over integers*. If they are all rational numbers, we call it a *polynomial over rational numbers*.

For example, $5x^2-2x+3$ is a polynomial over integers.

$\frac{2}{5}x^3+\frac{1}{3}x^2-\frac{3}{4}x-1$ is a polynomial over rationals.

$2x^3-\sqrt{3}x^2+x+2\sqrt{5}$ is a polynomial over reals.

A polynomial having a single term is called **monomial,** *e.g.*, $3x$, $\frac{4}{7}x^2$, $\sqrt{2}x^3$.

A polynomial having two terms is called a **binomial,** *e.g.*, $2x-5$, $3x^2+1$.

A polynomial having three terms is called a **trinomial,** *e.g.*, x^2-3x+2.

Note that the word '*monomial*' or '*binomial*' or '*trinomial*' is decided by the number of terms.

We write a polynomial in the *standard form* by arranging the terms in either ascending powers of x or descending powers of x.

Consider $p(x)=5x^3-2x+7-\frac{3}{4}x^2$.

It can be written in a standard form with powers of x in ascending order as

$$7-2x-\frac{3}{4}x^2+5x^3.$$

It can also be written in a standard form with powers of x in descending order as

$$5x^3-\frac{3}{4}x^2-2x+7.$$

The polynomial having all coefficients as zero is called zero polynomial.

For example, $0.x^2+0.x+0$ and $0.x^3-0.x+0$ are zero polynomials.

The exponent of the highest power of x appearing in a non-zero polynomial is called the degree of the polynomial

$5x+1$, $-3x+\frac{1}{2}$, $4x$ are the examples of **first degree** or **linear polynomials.**

A **linear** polynomial is of the form $ax+b$, where $a \neq 0$.

$3x^2-5x+1$, $5x^2-3$, x^2+6x-7 are the examples of **second degree** or **quadratic polynomials.**

A **quadratic** polynomial is of the form ax^2+bx+c, where $a \neq 0$.

Consider the polynomial $3x^5-x^3+\frac{1}{2}x^2+7$.

Observe that the term with the *highest power* of x is $3x^5$. Its exponent is 5. So the **degree** of the polynomial is 5.

Note that we do not assign any degree to the zero polynomial.

Terms involving the same power of the variable are called like terms.

In the polynomial $3-x^2+5x+7x^2$, terms $-x^2$ and $7x^2$ are like terms.

EXERCISE—5 (a)

(Section—A)

1. Which of the following functions are polynomials ?

(*a*) $\frac{3}{4}x-\frac{5}{8}$ (*b*) $3x^2-5x+\sqrt{2}$

(*c*) $\sqrt{2}x^3+x+\sqrt{5}$ (*d*) $\sqrt{x}+\sqrt{5}$

(*e*) $x+\frac{3}{x}$ (*f*) $3x^2+x^3-x-2$.

2. Classify each of the following polynomials as monomial, binomial or trinomial :

(*a*) $5x^2-8x+3$ (*b*) $7x$

(*c*) $2x+5$ (*d*) $3x^2-4x+\frac{2}{5}$

(*e*) $6x^4$ (*f*) $6x^3+4x-7$.

3. Write each of the following polynomials in ascending as well as descending order :

(*a*) $6a^2-3a-5+7a^3$ (*b*) $5x^3+6x^2-3x$

(*c*) $9x^2+8x^3+6x^4-5x+10$ (*d*) $1-6m^5$.

4. Write down the degree of each of the polynomials given below ;

(*a*) $3x^3+x^2-5$ (*b*) $5x^2-2x+1$

(*c*) $3-x^2-4x^3-x^4$ (*d*) $\frac{1}{3}x^2-3$.

5. In each of the polynomials given below, list the like terms :

(*a*) $5x^{10}-6x^2+7x^3-5x+7x^{10}-8x^2-9x-6x^3+1$

(*b*) $40-x+x^3-x^2+25-4x+50x^3-4x^4-5$

(*c*) $1-t+4t^3-6t^2+5t-8t^2-t^3$

(*d*) $y^4-y^3-y^2-y+1+y^4$.

5.2. ADDITION OF POLYNOMIALS

Let us revise the addition of polynomials.

(*a*) *Addition of monomials.*

Example 1. Add together :

$$5x, \quad 3x, \quad -4x, \quad 7x, \quad -2x.$$

Solution. Sum of positive numerical coefficients $=5+3+7$
$=15$

Sum of negative numerical coefficients $=4+2$
$=6$

$\therefore$ the difference $=15-6=9.$

Here the sign to be prefixed is positive.

$\therefore$ the required sum $=9x$.

(*b*) *Addition of binomials, trinomials, etc.*

The polynomials can be added by arranging like terms in vertical columns and then adding each column.

Example 2. Add the following polynomials :

$$\frac{1}{2}y^2+\frac{\sqrt{3}}{5}y-\sqrt{2}\,; \quad 2y^2+3-\sqrt{3}y.$$

Solution. Arranging the terms so that like terms are in one column

$$\begin{array}{llll} \frac{1}{2}y^2 & +\frac{\sqrt{3}}{5}y & -\sqrt{2} & \\ 2y^2 & -\sqrt{3}y & & +3 \\ \hline \end{array}$$

$$\left(\frac{1}{2}+2\right)y^2+\sqrt{3}\left(\frac{1}{5}-1\right)y+(3-\sqrt{2})$$

So, the sum is $\frac{5}{2}y^2-\frac{4}{5}\sqrt{3}y+(3-\sqrt{2})$.

Example 3. Find the sum of the following polynomials :

$$p(x)=x^4-5x^3-3x+1 \quad \text{and} \quad q(x)=7x^3+8x-4.$$

Solution. $p(x)+q(x)=(x^4-5x^3-3x+1)+(7x^3+8x-4)$

$$=x^4+(7x^3-5x^3)+(8x-3x)+(1-4)$$

$$=x^4+2x^3+5x-3.$$

EXERCISE—5 (b)

(Section—A)

Find the sum of :

1. $5x, 3x, 4x, x.$
2. $7x^2, 5x^2, x^2, 2x^2.$
3. $-13x, 7x, -8x, 5x.$
4. $\frac{3}{4}x^2, \frac{1}{2}x^2, -\frac{7}{4}x^2, \frac{3}{2}x^2.$

(Section—B)

Find the sum of the following polynomials :

5. $p(x)=x^4-3x^3+2x+6$ and $q(x)=x^3-3x+2.$
6. $p(x)=4x^3-3x^2+2$ and $q(x)=x^4-2x^3+6.$
7. $\frac{1}{\sqrt{2}}y^3+\frac{1}{\sqrt{3}}y-\frac{4}{5}$ and $\frac{8}{\sqrt{10}}y^2+\frac{2}{\sqrt{3}}y-\frac{1}{2}.$
8. $\frac{1}{\sqrt{3}}x-\frac{2}{\sqrt{3}}x^2+\frac{1}{3}x^3$ and $\sqrt{7}+\frac{1}{\sqrt{3}}x^2-1\cdot 2x^3.$

(Section—C)

Find the sum of :

9. $5x^2+2x+3, -4x^2+9, -6x-4.$
10. $x^3-5x^2+8, 5x^3-4x+7, -7x^3+6x^2+8x-5.$
11. Let $p(x)=x^3-5x^2+x-6$ and $q(x)=5x^2-x^3-x^4-2.$
    Find the degree of $p(x)+q(x).$
12. Let $p(x)=x^2-5x+3+x^4$

    $q(x)=5-$ [illegible] x^2+x^3+3x

    $r(x)=5x^2+2x-3x^5$

    Find the degree of $p(x)+q(x)+r(x).$

5.3. SUBTRACTION OF POLYNOMIALS

The polynomials can be subtracted by writing subtrahend under the minuend with like terms in the same vertical column and then changing the sign of each term of the subtrahend mentally and proceeding as in addition.

Example 1. Let $p(x)=x^2-8x+15$ and $q(x)=3x^2+17x-12.$ Find $q(x)-p(x).$

Solution. $q(x)-p(x)=3x^2+17x-12-(x^2-8x+15)$

$$=3x^2+17x-12-x^2+8x-15$$

$$=(3x^2-x^2)+(17x+8x)-(12+15)$$

$$=2x^2+25x-27.$$

Example 2. Subtract $6x^2-7$ from $8x^3-5x^2+6x-8$.

Solution.

$$\begin{array}{rl} \text{Minuend} & 8x^3-5x^2+6x-8 \\ \text{Subtrahend} & 6x^2-7 \\ \hline & 8x^3-11x^2+6x-1 \\ \hline \end{array}$$

In the first and third columns, as there is nothing to be subtracted, the terms are put down as such.

Example 3. Subtract $3x^2-2x$ from $1-x^3+x^4$.

Solution.

$$\begin{array}{rl} \text{Minuend} & x^4-x^3+1 \\ \text{Subtrahend} & 3x^2-2x \\ \hline & x^4-x^3-3x^2+2x+1 \\ \hline \end{array}$$

EXERCISE—5 (c)

(Section—A)

Perform the following subtractions :

1. $\begin{array}{r} 3x+7 \\ x+4 \\ \hline \end{array}$

2. $\begin{array}{r} 5x-9 \\ 2x+3 \\ \hline \end{array}$

3. $\begin{array}{r} 8x+11 \\ -3x-5 \\ \hline \end{array}$

4. $\begin{array}{r} 2x^2-x \\ x^2+9x \\ \hline \end{array}$

5. $\begin{array}{r} 4x^3+6x \\ -x^3-8x \\ \hline \end{array}$

6. $\begin{array}{r} 7x^4-5x^2 \\ -4x^4+3x^2 \\ \hline \end{array}$

(Section—B)

7. Subtract $x^2-8x+15$ from $3x^2+17x-12$.
8. Subtract $14x^2-5x+8$ from $13x^2-7$.
9. Subtract x^3-5x^2+7x-6 from $2x^3-4x^2+7x+5$.
10. Subtract x^3+2x^2-4x-1 from $4+5x+3x^3$.
11. Subtract $6x^2-3x+4$ from $2x^3$

(Section—C)

12. Let $p(x)=x^3-3x^2+7x-10$ and $q(x)=1-3x+4x^2-5x^3$. Find the degree of $p(x)-q(x)$.
13. If $p(x)=x^4-1+x-x^2$ and $q(x)=1-x+x^5-x^4-x^3$, find the degree of $q(x)-p(x)$.
14. Simplify :

$$\left(1+\frac{1}{2}x+\frac{1}{\sqrt{2}}x^2\right)+\left(5-\frac{2}{3}x+\sqrt{3}x^2\right)-\left(2+\frac{1}{9}x-\sqrt{6}x^2\right).$$

15. From $5x^3+3x-1$ take the sum of $2x-5+7x^2$ and $3x^2+4-2x^3+x$.

5.4. MULTIPLICATION OF POLYNOMIALS

(a) Multiplying monomials.

Example 1. Multiply $5x^3$ by $-9x^5$.

Solution.

$$\begin{aligned} 5x^3\times(-9x^5) &= 5\times(-9)\times x^3\times x^5 \\ &= -45\times x^{3+5} \\ &= -45x^8. \end{aligned}$$

Example 2. Find the product of $-3x$, $6x^2$ and $4x^3$.

Solution. $(-3x)\times(6x^2)\times(4x^3)=(-3)\times 6\times 4\times(x\times x^2\times x^3)$

$$=-72x^6$$

EXERCISE—5(d)

(Section—A)

Multiply :

1. $2x^2$ by $3x^3$.
2. $5x^4$ by $-9x^5$.
3. $4x^7$ by $-3x^5$.

(Section—B)

Find the product of :

4. $(-5x^2)$, $-4x^3$, $-3x^5$.
5. $\frac{1}{3}x^5$, $\frac{1}{4}x^3$, $-\frac{1}{9}x^2$.
6. $-3x^3$, $-\frac{4}{3}x^5$, $-\frac{1}{4}x^7$
7. $-\frac{3}{5}x^4$, $-\frac{2}{9}x^5$, $-\frac{1}{3}x^2$.
8. $\sqrt{2}x^2$, $\frac{1}{3}x^4$, $\frac{1}{\sqrt{2}}x$
9. $-2\sqrt{3}x^3$, $\frac{1}{6}x^2$, $\frac{1}{\sqrt{6}}x^4$

(*b*) *Multiplying a polynomial by a monomial.*

To multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add these products.

Example 3. Multiply $3x-5$ by $4x^2$.

Solution.

$$\begin{array}{r} 3x-5 \\ 4x^2 \\ \hline 12x^3-20x^2 \\ \hline \end{array}$$

$$3x\times 4x^2=3\times 4\times x\times x^2$$
$$=12x^3$$
$$-5\times 4x^2=(-5)\times 4\times x^2$$
$$=-20x^2.$$

Working of multiplication can also be arranged horizontally as under

$$4x^2\times(3x-5)=4x^2\times 3x-4x^2\times 5$$
$$=4\times 3\times x^2\times x-4\times 5\times x^2$$
$$=12x^3-20x^2$$

Example 4. Multiply $2x^4-3x^3+5x^2-7x+3$ by $-5x^2$.

Solution. $(2x^4-3x^3+5x^2-7x+3)\times(-5x^2)$

$$=2x^4\times(-5x^2)-3x^3\times(-5x^2)+(+5x^2)\times(-5x^2)-7x\times(-5x^2)+3\times(-5x^2)$$
$$=-10x^6+15x^5-25x^4+35x^3-15x^2.$$

EXERCISE—5(e)

(Section—A)

Multiply :

1. $3x-7$ by $5x^3$.
2. $5x^3-2x$ by $-3x^2$.
3. x^2-7x+8 by $3x^4$.
4. $3a^3-2a^2+4a-3$ by $5a^2$.
5. x^2-4x+9 by $-5x^2$.
6. $4a^2-6a+12$ by $\frac{1}{2}a^4$.

(Section—B)

Multiply

7. $5x^3-3x^2-1$ by $2x^3$

8. $5x^3-4x^2-3x+4$ by $-2x^2$.

9. $3x^4-\frac{2}{3}x^2+\frac{5}{6}x-7$ by $3x^2$.

10. $\sqrt{2}x^5-2x^3+\frac{1}{\sqrt{3}}x-5\log\frac{1}{\sqrt{2}}x^3$

(c) *Multiplying a polynomial by a polynomial.*

The product of two polynomials is found using the *distributive law.* However the method given below is more convenient.

We multiply a polynomial by another in the following steps :

(*i*) Arrange the terms of both polynomials in ascending or descending powers of one of the letters.

(*ii*) Starting from the left, multiply each term of the first polynomial by each term of the second.

(*iii*) Arrange like terms in same vertical column and add the products.

Example 5. If $p(x)=2x-5$ and $q(x)=x^2-2x+3$, find $p(x)$, $q(x)$.

Solution.

$$\begin{aligned} p(x)\,q(x)&=(2x-5)(x^2-2x+3)\\ &=2x(x^2-2x+3)-5(x^2-2x+3)\\ &=2x^3-4x^2+6x-(5x^2-10x+15)\\ &=2x^3-4x^2+6x-5x^2+10x-15\\ &=2x^3-4x^2-5x^2+6x+10x-15\\ &=2x^3-9x^2+16x-15 \end{aligned}$$

Example 6. Multiply $x^3-x+1+x^2$ by $1-x^2+x$.

Solution. Arranging both multiplicand and multiplier in ascending powers of x we have

$$\begin{array}{llllll|l} 1 & -x & +x^2 & +x^3 & & & \\ 1 & +x & -x^2 & & & & \\ \hline 1 & -x & +x^2 & +x^3 & & & \text{Multiplying by 1.} \\ & +x & -x^2 & +x^3 & +x^4 & & \text{Multiplying by } x. \\ & & -x^2 & +x^3 & -x^4 & -x^5 & \text{Multiplying by } -x^2. \\ \hline 1 & & -x^2 & +3x^3 & & -x^5 & \text{Adding the products.} \\ \hline \end{array}$$

EXERCISE—5(f)

(Section—A)

Multiply :

1. $2x+3$ by $3x-5$.
2. $3x+4$ by $2x-7$.
3. $4x^2-1$ by $5x-1$.
4. $5x^2+7$ by $3x+4$.
5. $4x^2-x+5$ by $5x+2$.
6. $4x^2-x+2$ by $5x-2$.

(Section—B)

Multiply :

7. $3x^2+4$ by $3x^2-4$.
8. $5x^2-9$ by $5x^2+9$.

9. If $p(x)=\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{3}}$ and $q(x)=x^2-\frac{1}{\sqrt{3}}x+\sqrt{2}$, find $p(x)\,.\,q(x)$.

10. If $p(x)=2x^2-5x+8$ and $q(x)=x^2-3x+7$, find $p(x)\,.\,q(x)$.

(Section—C)

11. Multiply $-6x+6+3x^2-x^3$ by $\frac{x^2}{2}+x+\frac{x^3}{6}+1$. Give the degree of the polynomial obtained as the product.

12. If $p(x)=\frac{3}{4}x^3-\frac{2}{3}x+\frac{1}{9}$ and $q(x)=\frac{1}{2}x^2+\frac{1}{3}x-1$, find $p(x)\,.\,q(x)$ and its degree.

5.3. DIVISION OF POLYNOMIALS

(*a*) *Division of monomials.*

Example 1. Divide $96x^7$ by $8x^4$.

Solution.

$$96x^7\div 8x^4=\frac{96x^7}{8x^4}$$

$$=\frac{96}{8}\times\frac{x^7}{x^4}$$

$$=12\times x^{7-4}\qquad =12x^3.$$

EXERCISE—5(g)

(Section—A)

Divide :

1. $12x^6$ by $2x^4$.
2. $-30x^9$ by $5x^5$.
3. $-24x^7$ by $16x^4$.
4. $35x^5$ by $28x^5$.
5. $28x^8$ by $-21x^6$.
6. $-15x^9$ by $-6x^8$.

(Section—B)

Find the quotient :

7. $\sqrt{3}x^7\div 2x^4$
8. $\sqrt{6}x^9\div 2\sqrt{3}x^5$.
9. $-7\sqrt{5}x^6\div\sqrt{5}x^2$.
10. $12\sqrt{6}x^{10}\div -3\sqrt{2}x^7$.

(*b*) *Division of a polynomial by a monomial.*

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial and take the sum of these partial quotients.

Example 2. Divide $8x^5-10x^2$ by $2x^2$.

Solution.

$$(8x^5-10x^2)\div 2x^2=\frac{8x^5-10x^2}{2x^2}$$

$$=\frac{8x^5}{2x^2}-\frac{10x^2}{2x^2}$$

$$=4x^3-5.$$

Example 3. Divide $12y^4-9y^3+6y^5$ by $-3y^3$.

Solution. $(12y^4-9y^3+6y^5)\div -3y^3$

$$=\frac{12y^4-9y^3+6y^5}{-3y^3}$$

$$=\frac{12y^4}{-3y^3}-\frac{9y^3}{-3y^3}+\frac{6y^5}{-3y^3}$$

$$=-4y+3-2y^2 \quad \textit{i.e.,}\ 3-4y-2y^2.$$

EXERCISE—5(h)

(Section—A)

Divide :

1. $8x^3-4x^2$ by $4x$.
2. $10x^4-25x^2$ by $5x$
3. $9x^4-6x^2+8x^3$ by $2x^2$.
4. $2x^4-4x^3+8x^2$ by $-2x$.

(Section—B)

Divide :

5. $\sqrt{2}x^4-6x^3-9x^2+15x$ by $-3x$.
6. $8x^5+\sqrt{3}x^3-\sqrt{2}x^2$ by $-5x^2$
7. $4\sqrt{3}x^6-\sqrt{6}x^5+9x^4-\sqrt{7}x^3$ by $\sqrt{3}x^3$.
8. $5\sqrt{2}x^4\times 3\sqrt{2}x^5-8x^6+6x^7$ by $-2\sqrt{2}x^4$.

(c) Division of a polynomial by a polynomial.

Let $p(x)$ and $q(x)$ be two polynomials such that $g(x)\neq 0$.

If we can find polynomials $q(x)$ and $r(x)$ such that

$$p(x)=g(x)\,.\,q(x)+r(x),$$

where degree $r(x)<$degree $g(x)$, then we say that $p(x)$ divided by $g(x)$ gives $q(x)$ as the *quotient* and $r(x)$ as the *remainder*.

If the remainder $r(x)$ is *zero*, we say that divisor $g(x)$ is a *factor* of $p(x)$.

To divide a polynomial by another, arrange both the dividend and division in ascending or descending powers of the variable. Then follow the method shown in the following examples :

Example 4. Divide $6x^3+x^2+5x+3$ by $2x+1$.

Solution,

$$
\begin{array}{r|l}
 & 3x^2-x+3 \\
\hline
2x+1 & 6x^3+x^2+5x+3 \\
 & 6x^3+3x^2 \\
\hline
 & -2x^2+5x \\
 & -2x^2-x \\
\hline
 & 6x+3 \\
 & 6x+3 \\
\hline
 & 0
\end{array}
$$

(i) $6x^3\div 2x\;=3x^2$

(ii) $-2x^2\div 2x=-x$

(iii) $6x\div 2x\;=3$

The required quotient$=3x^2-x+3$ and the remainder is zero.

So, $2x+1$ is a factor of $6x^3+x^2+5x+3$.

Example 5 Divide $6x^4-x^3+5x-6+4x^2$ by $3x^2-2+x$.

Solution. Arranging the dividend and divisor in the descending powers of x, we have

$$
\begin{array}{r|l}
 & 2x^2-x+3 \\
\hline
3x^2+x-2 & 6x^4-x^3+4x^2+5x-6 \\
 & 6x^4+2x^3-4x^2 \\
\hline
 & -3x^3+8x^2+5x \\
 & -3x^3-x^2+2x \\
\hline
 & 9x^2+3x-6 \\
 & 9x^2+3x-6 \\
\hline
 & 0
\end{array}
$$

(i) $6x^4\div 3x^2=2x^2$

(ii) $-3x^3\div 3x^2=-x$

(iii) $9x^2\div 3x^2=3$

The required quotient$=2x^2-x+3$ and the remainder is zero.

So, $3x^2-x-2$ is a factor of $6x^4-x^3+4x^2+5x-6$. We write it as

$$6x^4-x^3+4x^2+5x-6=(3x^2+x-2)(2x^2-x+3)$$

EXERCISE—5(i)

(Section—A)

Divide :

1. x^2+4x+4 by $x+2$.
2. $2x^2-x-15$ by $x-3$.
3. $x^2-3x-10$ by $x-5$.
4. $5x^2-17x+6$ by $5x-2$.
5. $6x^3+x^2+5x+3$ by $2x+1$.
6. $15+3a-7a^2-4a^3$ by $5-4a$,

(Section—B)

Divide

7. $x^4-4x^2+12x-9$ by x^2+2x-3.
8. $3x^4-14x^3+12x^2-26x-7$ by x^2-4x-1.
9. $x^4+57x-70$ by x^2+3x-5.

(Section—C)

10. Divide $p(x)$ by $g(x)$, where

$$p(x)=x^4+1 \text{ and } g(x)=x+1.$$

11. Divide $p(x)$ by $g(x)$ and find the quotient and the remainder, where

$$p(x)=2x^4+8x^3+14x^2+9x+2 \text{ and } g(x)=x^2+3x+1.$$

12. Divide $125x^6+127$ by $5x^2+3$ and find the quotient and the remainder.

5.6. PROPERTIES OF POLYNOMIALS

(*a*) Let $p(x)=3x^3-7x+4$ and $q(x)=-4x^2+5x-9$

$$\begin{aligned}p(x)+q(x)&=(3x^3-7x+4)+(-4x^2+5x-9)\\&=3x^3-4x^2+(-7+5)x+(4-9)\\&=3x^3-4x^2-2x-5\end{aligned}$$

$$\begin{aligned}q(x)+p(x)&=(-4x^2+5x-9)+(3x^3-7x+4)\\&=3x^3-4x^2+(5-7)x+(-9+4)\\&=3x^3-4x^2-2x-5.\end{aligned}$$

Therefore $\quad p(x)+q(x)=q(x)+p(x)$

(*b*) Let $\quad p(x)=5x^3-4x^2+x-3$ and $q(x)=0$

Now $\quad p(x)+q(x)=(5x^3-4x^2+x-3)+0$

$$=5x^3-4x^2+x-3 \qquad =p(x)$$

When a polynomial $p(x)$ is added to the zero polynomial, the sum is the polynomial $p(x)$ itself.

(*c*) Let $\quad p(x)=7x^2-9x+4$ and $q(x)=0$

Now $\quad p(x)-q(x)=(7x^2-9x+5)-0$

$$=7x^2-9x+4 \qquad =p(x)$$

When the zero polynomial is subtracted from a polynomial $p(x)$, the difference is the polynomial $p(x)$ itself.

(*d*) Let $p(x)=x^2+4x-1$ and $q(x)=x^2-x+3$

$$\begin{aligned}
p(x)\,.\,q(x)&=(x^2+4x-1)(x^2-x+3)\\
&=x^2(x^2-x+3)+4x(x^2-x+3)-1(x^2-x+3)\\
&=x^4-x^3+3x^2+4x^3-4x^2+12x-x^2+x-3\\
&=x^4+(-1+4)x^3+(3-4-1)x^2+(12+1)x-3\\
&=x^4+3x^3-2x^2+13x-3
\end{aligned}$$

$$\begin{aligned}
q(x)\,.\,p(x)&=(x^2-x+3)(x^2+4x-1)\\
&=x^2(x^2+4x-1)-x(x^2+4x-1)+3(x^2+4x-1)\\
&=x^4+4x^3-x^2-x^3-4x^2+x+3x^2+12x-3\\
&=x^4+(4-1)x^3+(-1-4+3)x^2+(1+12)x-3\\
&=x^4+3x^3-2x^2+13x-3
\end{aligned}$$

Therefore, $p(x)\,.\,q(x)=q(x)\,.\,p(x)$

(*e*) Let $p(x)=2x^3-x^2+4x-5$ and $q(x)=0$

Now
$$\begin{aligned}
p(x)\,.\,q(x)&=(2x^3-x^2+4x-5)\,0\\
&=(2x^3)\times 0-x^2\times 0+4x\times 0-5\times 0\\
&=0.
\end{aligned}$$

When a polynomial is multiplied by the zero polynomial, the product is the zero polynomial,

EXERCISE—5(j)

1. Let $p(x)=4x^3-5x^2+1$ and $q(x)=-5x^4+6x^3-3$.
   Verify that $p(x)+q(x)=q(x)+p(x)$.
2. Show that the sum of $p(x)$ and the zero polynomial is the polynomial $p(x)$ itself, where
   $p(x)=-5+6x^4-3x+4x^2$.
3. Let $p(x)=5x^2-x+1$ and $q(x)=3x^3-4x^2+5x-3$.
   Verify that $p(x)\,.\,q(x)=q(x)\,.\,p(x)$.
4. Show that the product of $p(x)$ and the zero polynomial is the zero polynomial, where
   $p(x)=-3x+5x^2-x^3+7$.
5. When the zero polynomial is subtracted from a polynomial $p(x)$, the remainder is the polynomial $p(x)$ itself, where
   $p(x)=5x^2-2x^3+x^4-7x+4$. Verify it.

5.7. REMAINDER THEOREM

We shall now discuss a method for finding the remainder without actually performing the process of division.

Let $p(x)=2x^2+3x-5$. We divide it by $x-2$ as under :

$$\begin{array}{r r}
 & 2x+7 \\
x-2) & \overline{2x^2+3x-6} \\
 & 2x^2-4x \\
 & \overline{\qquad 7x-5} \\
 & 7x-14 \\
 & \overline{\qquad\quad 9}
\end{array}$$

Here quotient $=2x+7$ and remainder $=9$.

Then $\qquad p(2)=2\times 2^2+3\times 2-5$

$\qquad\qquad =8+6-5 \qquad$ *i.e.*, 9

Thus we find that $p(2)$ is equal to the remainder when $p(x)$ is divided by $x-2$.

Consider $p(x)=x^4-2x^3+x-7$ to be divided by $x+3$.

$$
\begin{array}{rl}
 & x^3-5x^2+15x-44 \\ \cline{2-2}
x+3) & x^4-2x^3+x-7 \\
 & x^4+3x^3 \\ \cline{2-2}
 & \quad -5x^3+x \\
 & \quad -5x^3-15x^2 \\ \cline{2-2}
 & \qquad\quad 15x^2+x \\
 & \qquad\quad 15x^2+45x \\ \cline{2-2}
 & \qquad\qquad -44x-7 \\
 & \qquad\qquad -44x-132 \\ \cline{2-2}
 & \qquad\qquad\quad +125
\end{array}
$$

Here quotient $=x^3-5x^2+15x-44$ and remainder $=125$

We know that $p(x)=g(x)\,.\,q(x)+r(x)$.

Then $(x+3)\times q(x)+r(x)=x^4-2x^3+x-7$.

Putting $x=-3$. we get

$$(-3+3)\times q(x)+r(x)=(-3)^4-2(-3)^3+(-3)-7$$

$\Rightarrow \qquad 0\times q(x)+r(x)=81+54-3-7$

$\therefore \qquad r(x)=125$

In fact $p(-3)$ is equal to 125 which is the remainder.

Note that the same remainder is obtained by both methods.

When a polynomial $p(x)$ is divided by $x-a$, the remainder is given by $p(a)$.

$p(x)$ is *divisible* by $x-a$, when $p(a)=0$

Similarly $p(x)$ is *divisible* by $x+a$, when $p(-a)=0$

We can now state the **remainder theorem** as follows :

Let $p(x)$ be any polynomial of degree $\geqslant 1$ and a any real number. If $p(x)$ is divided by $x-a$, then the remainder is $p(a)$.

Let $p(x)$ be a polynomial of degree $n>0$. If $p(a)=0$ for a real number a, then $(x-a)$ is a factor of $p(x)$.

Conversely, if $(x-a)$ is a factor of $p(x)$, then $p(a)=0$.

This is known as **factor theorem**.

Example 1. Find the remainder when $x^4+2x^3-3x^2+2x-5$ is divided by $x-2$.

Solution. Here $p(x)=x^4+2x^3-3x^2+2x-5$

$\therefore \qquad p(2)=2^4+2\times 2^3-3\times 2^2+2\times 2-5$

So, the remainder $=p(2)=16+16-12+4-5=19$

Hence the required remainder $=19$.

Example 2. Determine if $x+3$ is a factor of $x^3+8x^2+19x+12$.

Solution. $x+3=x-(-3)$

Let $p(x)=x^3+8x^2+19x+12$

Then $p(-3)=(-3)^3+8(-3)^2+19(-3)+12$

$=-27+72-57+12=0$

$\therefore\quad p(-3)=0$

Since $p(-3)=0$, $x-(-3)$ *i.e.*, $x+3$ is a factor of $p(x)$ by the factor theorem.

EXERCISE—5(k)

(Section—A)

1. If $p(x)=x^3-6x^2+11x-5$, find the remainder when $p(x)$ is divided by
   (*a*) $x-1$ (*b*) $x-2$
2. Let $p(x)=6x^4+8x^3+3x^2+5x+3$, find the remainder when $p(x)$ is divided by
   (*a*) $x+1$ (*b*) $x+2$
3. Find the remainder when $x^3+x^2-14x-29$ is divided by $x+3$.
4. Find the remainder when $x^3-2x^2-23x-80$ is divided by $x+5$.
5. Show that $3x^3-2x^2-7x-2$ is divisible by $x-2$.
6. Prove that $x+1$ is a factor of $2x^4-5x^3+5x-2$.

(Section—B)

7. Show that $x-2$ and $x+4$ are the factors of $2x^3+7x^2-10x-24$.
8. For what value of m will $x-2$ be a factor of x^3-5x^2+mx+m ?
9. Find the value of a for which $x+1$ may be a factor of $2x^3-ax^2-(2a-3)x+2$.
10. Find the value of a, if $x-a$ is a factor of x^3-a^2x+x+2.

(Section—C)

11. If $x-4$ and $x+6$ are the factors of $3x^3+ax^2-74x+b$, find the values of a and b.
12. The polynomials px^3+2x^2-3 and x^4-px+4 leave the same remainder when they are divided by $x-2$. Find the value of p.
13. Find the values of p and q so that $(x+1)$ and $(x-1)$ are factors of

$$x^4+px^3+2x^3-3x+q.$$

14. What number should be (*a*) added to, (*b*) subtracted from $4x^3-6x^2+2x+7$ to make it divisible by $2x+1$?

REVIEW EXERCISE—V

(Section—A)

1. Fill in the blanks to make the following statements true :
   (a) A polynomial has only...... powers of the variable in its terms.
   (b) A polynomial having ..... terms is called trinomial.
   (c) All the coefficients in a zero polynomial are .. ...
   (d) The degree of the polynomial -7 is ....
   (e) If $(x-a)$ is a factor of $p(x)$, then $p(a)$ equals ....
   (f) The domain of a polynomial over reals is the set of ... ..
   (g) A polynomial of degree n has at the most ..... terms.
2. Which of the following functions are polynomials ?
   (a) $\sqrt{2}\, y^3+\sqrt{3}y$ (b) $\sqrt{2}\, \sqrt{x}+\sqrt{3}x$
3. Let $p(x)=x^7-x^5+2x^2+1$ and $q(x)=u-u^7-2$. Find $p(x)+q(x)$.
4. Let $p(x)=3x^7-2x^3+3$ and $q(x)=x^6-3x^4+x^2+x$. Find $p(x)-q(x)$.
5. If $p(x)=1+x^2+x$ and $q(x)=x^2+1-x$, find $p(x).q(x)$.
6. If $p(x)=4x^3+7x^2-3x-15$ and $q(x)=4x-5$, find $p(x)\div q(x)$.

(Section—B)

7. If $p(x)=7x-x^2+2-x^3$ and $q(x)=5x^3-x^2+4+x^4$, find the degree of $p(x)+q(x)$.
8. If $p(x)=x^4-6x^2+9x-5$ and $q(x)=4x-3x^2-x^3+x^4$, find the degree of $p(x)-q(x)$.
9. If $p(x)=x^3-6x^2+x-3$ and $q(x)=x^2-3x+2$, find the degree of $p(x).q(x)$.
10. Divide $1-a^4$ by $1-a$.
11. Add together : $2+5x^2$, $-x-x^2-1$, x^2+2x+1 and $6x-2x^2$.
12. Subtract $2x^3+3$ from $5x^3+3x^2+7x-3$.
13. Multiply $5x^2-x+1$ by $3x^3-4x^2+5x-3$.
14. Divide $6x^3-x^2-22x+15$ by $2x-3$.
15. Find the remainder when $8x^3-2x^2-15x-9$ is divided by $4x-3$.

(Section—C)

16. Divide $2x^5+3x^2-3x-2$ by $x-1$.
17. Divide $x^{12}-2+x^6$ by $1+x^2+x^4$.
18. Find the remainder when $27x^3-9x^2+6x+4$ is divided by $3x+1$, using remainder theorem.
19. Divide $x^4-3x^3-13x^2+12x+4$ by x^2-x+2 and find the quotient and the remainder.
20. What must be added to $5x^4-7x^3+6$ so that the result may be divisible by $x+3$?
21. If $x+2$ and $x-3$ are the factors of $2x^3+3x^2+px+q$, find the value of p and q.

□ □

6

FACTORISING POLYNOMIALS

6.1. FACTORS

If $g(x)$ and $h(x)$ are two polynomials and their product is a polynomial $p(x)$, then we write as

$$g(x) \cdot h(x)=p(x)$$

Viewed the other way about, $g(x)$ and $h(x)$ are two **factors** of $p(x)$ and we write as

$$p(x)=g(x)\,h(x)$$

Thus, $p(x)$ can be factorised into two factors $g(x)$ and $h(x)$.

Since $4x+2$ is the product of two polynomials 2 and $2x+1$. So, 2 and $2x+1$ are *factors* of polynomial $4x+2$.

6.2. FACTORISING QUADRATIC POLYNOMIALS

Let us recall the following formulae learnt in earlier classes :

(*i*) $(x+a)^2=x^2+2ax+a^2$

(*ii*) $(x-a)^2=x^2-2ax+a^2$

(*iii*) $(x+a)(x-a)=x^2-a^2$

We know that $(a+b)^2=a^2+2ab+b^2$

When we multiply $(a+b)$ by $(a+b)$, we get $a^2+2ab+b^2$. Hence the factors of

$a^2+2ab+b^2$ are $(a+b)$ and $(a+b)$

$a^2+2ab+b^2=(a+b)(a+b)$ or $(a+b)^2$

Similarly we can write

$a^2-2ab+b^2=(a-b)^2$.

Example 1. Factorise $x^2+22x+121$.

Solution. $x^2+22x+121=(x)^2+2(11)(x)+(11)^2$
$=(x+11)^2$.

Example 2. Factorise $25x^2+40xy+16y^2$.

Solution. $25x^2+40xy+16y^2=(5x)^2+2(5x)(4y)+(4y)^2$
$=(5x+4y)^2$.

Example 3. Factorise $49x^2-42xy+9y^2$.

Solution. $49x^2-42xy+9y^2=(7x)^2-2(7x)(3y)+(3y)^2$
$=(7x-3y)^2$.

EXERCISE—6(a)

Factorise each of the following :

(Section—A)

1. $x^2+14x+49$.
2. $x^2-18x+81$.
3. $1-6m+9m^2$.
4. $4p^2+4p+1$.
5. $4x^2+12x+9$.
6. $9x^2-42x+49$.
7. $25x^2-40x+16$.
8. $16x^2+72x+81$.
9. $4x^2+12xy+9y^2$.
10. $9x^2-42xy+49y^2$.
11. $16x^2-120xy+225y^2$.
12. $49a^2-126ab+81b^2$.

(Section—B)

13. $4(m+n)^2+28(m+n)p+49p^2$.
14. $(a+b)^2-2(a+b)(a-b)+(a-b)^2$.

(Section—C)

15. $16a^2b^2c-8abc^3+c^5$.
16. $45a^2b+5ab^3-30a^2b^2$.

6.3. FACTORISING DIFFERENCE OF TWO SQUARES

We know that $(a+b)(a-b)=a^2-b^2$.

Therefore $(a+b)$ and $(a-b)$ are the factors of a^2-b^2.

$$a^2-b^2=(a+b)(a-b).$$

Example 1. Factorise $9x^2-16$.

Solution. $9x^2-16=(3x)^2-(4)^2$.

$$=(3x+4)(3x-4).$$

Example 2. Factorise $49x^2-81y^2$.

Solution. $49x^2-81y^2=(7x)^2-(9y)^2$

$$=(7x+9y)(7x-9y).$$

Example 3. Factorise x^4-16y^4.

Solution. $x^4-16y^4=(x^2)^2-(4y^2)^2$

$$\begin{aligned}&=(x^2+4y^2)(x^2-4y^2)\\&=(x^2+4y^2)[(x)^2-(2y)^2]\\&=(x^2+4y^2)(x+2y)(x-2y).\end{aligned}$$

Example 4. Factorise $49(x+y)^2-36(x-y)^2$.

Solution. $49(x+y)^2-36(x-y)^2=[7(x+y)]^2-[6(x-y)]^2$

$$\begin{aligned}&=(7x+7y)^2-(6x-6y)^2\\&=[(7x+7y)+(6x-6y)]\ [(7x+7y)-(6x-6y)\\&=(13x+y)(x+13y).\end{aligned}$$

EXERCISE—6(b)

(Section—A)

Factorise :

1. $64x^2-169y^2$.
2. $121a^2-36b^2$.
3. $1-81p^4q^2$.
4. $c^2-(a+b)^2$.
5. $16a^2-9b^2$.
6. $36x^2-\frac{1}{49}$.

(Section—B)

7. $9(a+b)^2-1.$

8. $1-25(x-y)^2.$

9. $(3x-1)^2-16x^2.$

10. $(6x+1)^2-(5x+1)^2.$

11. $81(x+2y)^2-16(2x-y)^2.$

12. $4(x-b)^2-(2x+a)^2.$

13. $25(x-y)^2-4(x+4)^2.$

14. $8-2x^2.$

15. $2-50x^2.$

16. $8x^3-2x.$

17. $16y^3-4y.$

18. $18a^3-8ab^2.$

(Section—C)

19. $16a^4-b^4.$

20. $x^4y^4-z^4.$

21. $x^2-y^2+2yz-z^2.$

22. $25x^2-10x+1-36y^2$

Example. Factorise $x^2-y^2-4xz+4z^2.$

Solution. $x^2-y^2-4xz+4z^2.$

$$
\begin{aligned}
&=(x^2-4xz+4z^2)-y^2\\
&=[(x)^2-2.x.2z+(2z)^2]-y^2\\
&=(x-2z)^2-(y)^2\\
&=(x-2z+y)(x-2z-y)\\
&=(x+y-2z)(x-y-2z).
\end{aligned}
$$

EXERCISE—6(c)

Factorise the following polynomials :

(Section—A)

1. $a^2-b^2+c^2+2ac.$

2. $x^2+y^2-z^2+2xy.$

3. $1-x^2-8xy-16y^2.$

4. $4x^2-y^2-9z^2+6yz.$

5. $a^2-b^2+2bc-c^2.$

6. $9x^2-4a^2+4ay-y^2.$

(Section—B)

7. $a^2+b^2+2ab-c^2-d^2+2cd.$

8. $p^2+q^2-r^2-2pq-2r-1.$

9. $2(ab+cd)-a^2-b^2+c^2+d^2.$

10. $(a^2-b^2)(c^2-d^2)-4abcd.$

We will now factorise those polynomials which can be reduced to the form a^2-c^2 by completing the square.

Example 1. Factorise $x^4+x^2y^2+y^4.$

Solution. $x^4+x^2y^2+y^4$

$$
\begin{aligned}
&=(x^2)^2+2.x^2.y^2+(y^2)^2-x^2y^2\\
&=(x^2+y^2)^2-(xy)^2\\
&=(x^2+y^2+xy)(x^2+y^2-xy)\\
&=(x^2+xy+y^2)(x^2-xy+y^2).
\end{aligned}
$$

Example 2. Factorise $x^4+4.$

Solution. x^4+4

$$
\begin{aligned}
&=(x^2)^2+(2)^2\\
&=(x^2)^2+2(x)^2.(2)+(2)^2-4x^2\\
&=(x^2+2)^2-(2x)^2\\
&=(x^2+2+2x)(x^2+2-2x)\\
&=(x^2+2x+2)(x^2-2x+2).
\end{aligned}
$$

EXERCISE—6(d)

Factorise :

(Section—A)

1. x^4+x^2+1.
2. x^4-11x^2+25.
3. $9x^4-7x^2+1$.
4. $x^4-x^2y^2+16y^4$.

(Section—B)

5. $4x^4+1$.
6. x^4+4y^4.
7. $4x^4+81$.
8. $81x^4+64y^4$.

6.4. FACTORISING TRINOMIALS : $x^2 \pm bx \pm c$

We know that the product of two linear polynomials is a *quadratic* polynomial.

Is a quadratic polynomial always the product of two linear polynomials ?

In case of some quadratic polynomials. it is possible to express the quadratic-polynomial as the product of linear polynomials.

Let us examine the following cases :

(*i*) $(x+1)(x+2)=x^2+3x+2$
(*ii*) $(x+4)(x-3)=x^2+x-12$
(*iii*) $(x-5)(x+3)=x^2-2x-15$
(*iv*) $(x-3)(x-4)=x^2-7x+12$.

In (*i*) we have $2=1\times 2$ and $1+2=3$
In (*ii*) we have $-12=(+4)(-3)$ and $(+4)+(-3)=1$
In (*iii*) we have $-15=(-5)(+3)$ and $(-5)+(3)=-2$
In (*iv*) we have $12=(-3(-4)$ and $(-3)+(-4)=-7$

What do you infer ?

We decompose the *constant term* of the trinomial into two factors such that their *sum* is equal to the *coefficient* of x.

Example 1. Factorise $x^2+13x+40$.

Solution. $x^2+13x+40$

$=x^2+5x+8x+40$
$=(x^2+5x)+(8x+40)$
$=x(x+5)+8(x+5)$
$=(x+5)(x+8)$

The possible factors are

$40=1\times 40$ $40=4\times 10$
$40=2\times 20$ $40=5\times 8$
$\mathbf{40=5\times 8}$; $\mathbf{5+8=13}$

EXERCISE—6 (e)

Factorise the following polynomials :

1. x^2+5x+6.
2. $x^2+8x+15$.
3. $x^2+7x+12$.
4. $x^2+11x+24$.
5. $x^2+19x+60$.
6. $x^2+20x+91$.
7. $x^2+19x+70$.
8. $x^2+27x+180$.
9. $x^2+28x+187$.
10. $x^2+35x+286$.

Example 2. Factorise $x^2-18x+65$.

Solution. $x^2-18x+65$

$=x^2-13x-5x+65$

$=(x^2-13x)-(5x-65)$

$=x(x-13)-5(x-13)$

$=(x-13)\,(x-5)$

$65=(-13)\times(-5)$

$(-13)+(-5)=-18$

EXERCISE—6 (f)

Factorise :

1. x^2+6x+8.
2. $x^2-8x+15$.
3. x^2-5x+6.
4. $x^2-7x+12$.
5. $x^2-14x+45$.
6. $x^2-25x+84$.
7. $x^2-22x+120$.
8. $x^2-27x+72$.
9. $x^2-27x+176$.
10. $x^2-39x+224$.

Example 3. Factorise $x^2+3x-40$.

Solution. $x^2+3x-40$

$=x^2+8x-5x-40$

$=(x^2+8x)-(5x+40)$

$=x(x+8)-5(x+8)$

$=(x+8)\,(x-5)$

$-40=(+8)\times(-5)$ and

$(+8)+(-5)=+3$

Example 4. Factorise $x^2-17x-84$.

Solution. $x^2-17x-84$

$=x^2-21x+4x-84$

$=(x^2-21x)+(4x-84)$

$=x(x-21)+4(x-21)$

$=(x-21)\,(x+4)$

$-84=(-21)\,(+4)$ and

$(-21)+(+4)=-17$

EXERCISE—6 (g)

Factorise :

1. x^2+x-6.
2. x^2-x-6.
3. $x^2+2x-15$.
4. $x^2-2x-15$.
5. $x^2+4x-45$.
6. $x^2-4x-45$.
7. $x^2+16x-105$.
8. $x^2-3x-108$.
9. $x^2+11x-126$.
10. $x^2-3x-130$.

6.5. FACTORISING TRINOMIALS : $ax^2\pm bx\pm c$

Let us factorise the trinomial ax^2+bx+c.

x^2+bx+c is a special form of ax^2+bx+c, where $a=1$.

We multiply the *coefficient* of x^2 and the *constant term*. Then we decompose this product into two factors such that their *sum* is equal to the *coefficient* of x.

Example 1. Factorise $2x^2-7x+6$.

Solution. $2x^2-7x+6$

$=2x^2-4x-3x+6$

$=(2x^2-4x)-(3x-6)$

$=2x(x-2)-3(x-2)$

$=(x-2)\,(2x-3)$

Coefficient of $x^2=2$

Constant term$=6$

Their product$=2\times6=12$

$12=(-4)\,(-3)$ and

$(-4)+(-3)=(-7)$

EXERCISE—6 (h)

Factorise :

1. $4x^2+12x+5$.
2. $6x^2+x-2$.
3. $2x^2+5x-3$.
4. $2x^2-x-6$.
5. $4x^2+13x-12$.
6. $6x^2-x-2$.
7. $6x^2-13x+5$.
8. $6x^2-x-12$.
9. $14x^2-25x+6$.
10. $12x^2+17x-40$.
11. $6x^2+13x-5$.
12. $8x^2+43x+15$.
13. $3x^2-4x-7$.
14. $6x^2-7x-3$.
15. $1-18y-63y^2$.
16. $40+3x-x^2$.

Example 2. Factorise $72x^2-53xy+5y^2$.

Solution. $72x^2-53xy+5y^2$

$$\begin{aligned}&=72x^2-8xy-45xy+5y^2\\&=(72x^2-8xy)-(45xy-5y^2)\\&=8x(9x-y)-5y(9x-y)\\&=(9x-y)(8x-5y)\end{aligned}$$

$$\begin{aligned}72\times5&=360\\360&=(-8)(-45)\\(-8)+(-45)&=-53\end{aligned}$$

EXERCISE—6 (i)

Factorise.

(Section—A)

1. $x^2+14xy+45y^2$.
2. $x^2+4xy-45y^2$.
3. $x^2-32xy+112y^2$.
4. $x^2-11xy-80y^2$.
5. $6x^2-11xy-10y^2$.
6. $8x^2-22xy+15y^2$.
7. $6x^2-5xy-6y^2$.
8. $3-4x-7x^2$.
9. $12x^2+17xy-40y^2$.
10. $20x^2-60xy+25y^2$.

(Section—B)

11. $x^2y^2-8xy-48$.
12. $a^2x^2-2abxy-15b^2y^2$.
13. $(x+4)^2-5(x+4)-6$.
14. $16x^4+36x^2+81$.

The following example illustrates factorisation by the method of completing the square.

Example 3. Factorise $x^2+2x-399$.

Solution. $x^2+2x-399=(x^2+2x+1)-1-399$

$$\begin{aligned}&=(x+1)^2-400\\&=(x+1)^2-(20)^2\\&=(x+1+20)(x+1-20)\\&=(x+21)(x-19)\end{aligned}$$

EXERCISE—6 (j)

Factorise the following polynomials by the method of completing the square :

1. $x^2+6x-72$.
2. $x^2+16x+55$.
3. $x^2+5x-24$.
4. $x^2-9x+20$.
5. x^2-2x-1.
6. x^2-4x+1.

6.6. FACTORISING $\mathbf{a^3+b^3}$

We know that $(a+b)^3=a^3+b^3+3ab(a+b)$

$\Rightarrow \quad a^3+b^3=(a+b)^3-3ab(a+b)$

Also $\quad a^3+b^3=(a+b)\,[(a+b)^2-3ab]$

$\quad =(a+b)\,[a^2+2ab+b^2-3ab]$

$\therefore \quad \mathbf{a^3+b^3=(a+b)\,(a^2-ab+b^2)}$

Example. **Factorise** $8a^3+27b^3$.

Solution. $8a^3+27b^3=(2a)^3+(3b)^3$

$=(2a+3b)\,[(2a)^2-(2a)\,(3b)+(3b)^2]$

$=(2a+3b)\,(4a^2-6ab+9b^2)$

EXERCISE—6 (k)

Resolve into factors :

(Section—A)

1. a^3+64.

2. a^2b^3+216.

3. $64x^3+1000$.

4. $1a^3+125b^3$.

(Section—B)

5. $27x^6+125y^6$.

6. $343a^4b-64ab^4$.

7. $(2a-3b)^3+27c^3$.

8. $(a-2b)^3+1$.

(Section—C)

9. $27(x+y)^3+8(2x-y)^3$.

10. $8(x-y)^3+125(y-2x)^3$.

6.7. FACTORISING $\mathbf{a^3-b^3}$

We know that $(a-b)^3=a^3-b^3-3ab\,(a-b)$

$\Rightarrow \quad a^3-b^3=(a-b)^3+3ab\,(a-b)$

Also $\quad a^3-b^3=(a-b)\,[(a-b)^2+3ab]$

$\quad =(a-b)\,[a^2-2ab+b^2+3ab]$

$\therefore \quad \mathbf{a^3-b^3=(a-b)\,(a^2+ab+b^2)}$

Example. Factorise $125x^3-8y^3$.

Solution. $125x^3-8y^3=(5x)^3-(2y)^3$

$=(5x-2y)\,[(5x)^2+(5x)\,(2y)+(2y)^2]$

$=(5x-2y)\,(25x^2+10xy+4y^2)$.

EXERCISE—6 (l)

Resolve into factors :

(Section—A)

1. $27x^3-1$.

2. $8-x^3$.

3. $27a^3-125b^3$.

4. $729x^3-216$.

(Section—B)

5. $8x^3-1000y^3$.

6. a^6-8b^6.

(Section—C)

7. a^7-27ab^6.

8. a^9-b^9.

9. $(3x+2y)^3-z^3$.

10. $125(a-2b)^3-27(3a-4b)^3$.

68. APPLICATION OF FACTOR THEOREM

We can factorise polynomials of 3rd or 4th degree using factor theorem. Study the worked out example given below :

Let us factorise $x^3-7x^2+14x-8$.

Let $f(x)=x^3-7x^2+14x-8$.

Since 8 is divisible by 1, 2, 4, 8, we will substitute 1, 2, 4, 8 for x.

Since it is a polynomial of 3rd degree, it should have *three linear factors.*

$f(1)=1-7+14-8=0$ Hence $x-1$ is a factor of $f(x)$.

$f(-1)=-1-7-14-8=-30$ Hence $x+1$ is not a factor.

$f(2)=8-28+28-8=0$ Hence $x-2$ is a factor.

$f(-2)=-8-28-28-8=-72$ Hence $x+2$ is not a factor.

$f(4)=64-112-56-8=0$ Hence $x-4$ is a factor.

Therefore $x^3-7x^2+14x-8=(x-1)(x-2)(x-4)$.

Example. Factorise $4x^3-4x^2-21x-9$

Solution. Let $f(x)=4x^3-4x^2-21x-9$

Since 9 is divisible by 1, 3, 9, we should substitute 1, 3, 9 for x.

Then $f(1)=4-4-21-9=-30$ So, $x-1$ is not a factor.

$f(-1)=-4-4+21-9=4$ So, $x+1$ is not a factor.

$f(3)=4\times27-4\times9-21\times3-9=108-36-63-9=0$

So, $x-3$ is a factor of $f(x)$.

$$\begin{aligned}4x^3-4x^2-21x-9&=4x^3-12x^2+8x^2-24x+3x-9\\&=4x^2(x-3)+8x(x-3)+3(x-3)\\&=(x-3)(4x^2+8x+3)\\&=(x-3)[4x^2+2x+6x+3]\\&=(x-3)[2x(2x+1)+3(2x+1)]\\&=(x-3)(2x+1)(2x+3)\end{aligned}$$

EXERCISE—6 (m)

(Section—A)

By means of the factor theorem, find the factors of the following polynomials :

1. $x^3-6x^2+11x-6$.
2. x^3-2x^2-5x+6.
3. x^3-2x^2-x+2
4. $x^3+x^2-10x+8$.
5. $x^3-3x^2-10x+24$.
6. $x^3+8x^2+19x+12$.
7. $x^3+7x^2+14x+8$.
8. $x^3+9x^2+23x+15$.

(Section—B)

Resolve into factors using the factor theorem :

9. $3x^3+x^2+x-5$.
10. $4x^3+3x^2+x+2$.
11. $6x^3+7x^2-16x-12$.
12. $2x^3-3x^2-12x+20$.

(Section—C)

Factorise the following polynomials using the factor theorem :

13. x^4-2x^3-6x-9.
14. $x^4-27x^2-14x+120$.
15. x^4-27x^2+14x.
16. $2x^4+x^3-14x^2-19x-6$.
17. $3x^4+4x^3-57x^2-18x-40$.
18. $2x^4+17x^3+23x^2-105x-225$.

6.9. H.C.F. OF POLYNOMIALS

We have already learnt how to find H.C.F. and L.C.M. of two numbers in earlier classes. The concept of H.C.F. and L.C.M. can be extended to polynomials also.

A polynomial $h(x)$ is called to highest common factor (H.C.F.) of two non-zero polynomials $f(x)$ and $g(x)$, if

(*i*) $h(x)$ is a common factor of $f(x)$ and $g(x)$.

(*ii*) the degree of $h(x)$ is highest among the common factors $f(x)$ and $g(x)$.

We will illustrate the method of finding H.C.F. through solved examples given below.

Example 1. Find the H.C.F. of x^2-4x+3 and x^2-5x+6.

Solution. Let us factorise the given polynomials :

$$\begin{aligned} &x^2-4x+3\\ &=x^2-x-3x+3\\ &=x(x-1)-3(x-1)\\ &=(x-1)(x-3)\end{aligned} \qquad \begin{aligned} &x^2-5x+6\\ &=x^2-2x-3x+6\\ &=x(x-2)-3(x-2)\\ &=(x-2)(x-3)\end{aligned}$$

We find that $(x-3)$ is a *common factor* of given polynomials. Also the only common factor of given polynomials is $(x-3)$.

Therefore, the required H.C.F. is $(x-3)$.

Example 2. Find the H.C.F. of $4x^2-16$ and $6x^2+24x+24$.

Solution. Factorising given polynomials, we have

$$4x^2-16=4(x^2-4) \quad =4(x+2)(x-2)$$

$$6x^2+24x+24=6(x^2+4x+4)=6(x+2)^2$$

The H.C.F. of the numerical factors 4 and 6 is 2.

The highest power common factor is $x+2$.

Hence, the H.C.F.$=2(x+2)$.

We find the H.C.F. of two non-zero polynomials in the following steps :

(1) *We express the polynomials $f(x)$ and $g(x)$ in the complete factored form i.e., as a product of numerical factor, linear factors and quadratic factors, if linear factors are not possible.*

(2) *We find the H.C.F. of the numerical factors of $f(x)$ and $g(x)$.*

(3) *We find the factors with highest power common to $f(x)$ and $g(x)$.*

(4) *We then find the product of all such common factors with highest power and the H.C.F. of the numerical factors.*

(5) *The product so obtained is the H.C.F. of $f(x)$ and $g(x)$.*

Sometimes it is easier to find the H.C.F. by division method. We shall now solve an example to illustrate this method.

Example 3. Find the H.C.F. of x^3+3x^2-6x-8 and $x^2+7x-18$.

Solution. We divide x^3+3x^2-6x-8 by $x^2+7x-18$ because x^3+3x^2-6x-8 is of higher degree.

$$\begin{array}{r|l} & x-4 \\ \hline x^2+7x-18 & x^3+3x^2-6x-8 \\ & x^3+7x^2-18x \\ \hline & -4x^2+12x-8 \\ & -4x^2-28x+72 \\ \hline & 40x-80 \end{array}$$

The remainder is $40x-80$. We divide the remainder by numerical factor 40 which is common to both the terms in the remainder. Thus, we get $x-2$.

Now the new divisor becomes $x-2$ and the earlier divisor $x^2+7x-18$ becomes the new dividend.

$$
\begin{array}{r|l}
 & \quad x+9 \\ \cline{2-2}
x-2 & x^2+7x-18 \\
 & x^2-2x \\ \cline{2-2}
 & \quad 9x-18 \\
 & \quad 9x-18 \\ \cline{2-2}
 & \qquad 0
\end{array}
$$

Since the remainder is 0, the process of division stops. The last divisor is $x-2$ which is the required H.C.F. of the given polynomials.

EXERCISE—6(n)

(Section—A)

Find the H.C.F. of the following polynomials :

1. $2x-4$ and $3x-6$
2. $3x+3$ and $5x+5$
3. x^2-x and $(x-1)^2$
4. $x+2$ and x^2-4
5. x^2-6x+8 and x^2-5x+6
6. $x^2+7x-18$ and $x^2+10x+9$
7. $2x^2+5x-42$ and $3x^2+13x-30$
8. $3x^2+5x-12$ and $15x^2-11x-12$

(Section—B)

Find the H.C.F., by factorisation method, of the following polynomials :

9. x^2-6x+9 and $x^3-3x^2-9x+27$
10. x^4-1 and x^3+x^2+x+1
11. x^3-5x^2+3x+1 and $x^3-6x^2+11x-6$
12. $x^3+2x^2-19x-20$ and $x^3+9x^2+23x+15$

(Section—C)

Find by the method of division the H.C.F. of the following polynomials :

13. x^3+2x^2-3 and x^3+x^2-2
14. x^3-2x-3 and $x^3+4x^2-7x-10$
15. $x^3-19x+30$ and $x^3-10x^2+31x-30$
16. $x^3+7x^2+15x+9$ and $x^3+6x^2+10x+3$
17. $3x^3-14x^2+9x+10$ and $15x^3-34x^2+21x-10$
18. $2x^3+20x^2+62x+60$ and $2x^3+18x^2+52x+48$

6.10. L.C.M. OF POLYNOMIALS

A polynomial $m(x)$ is called the least common multiple (L.C.M.) of two non-zero polynomials $f(x)$ and $g(x)$, if

(*i*) $m(x)$ is multiple of both $f(x)$ and $g(x)$,

(*ii*) the degree of $m(x)$ is the least among the common multiples.

We will illustrate the method of finding L.C.M. through solved examples given below :

Example 1. Find the L.C.M. of x^2-4 and x^2-4x+4.

Solution. First we factorise given polynomials.

$$x^2-4=(x+2)(x-2)$$

$$x^2-4x+4=x^2-2.2.x+(2)^2 \quad =(x-2)^2$$

Hence L.C.M. $=(x+2)(x-2)^2$

$$=(x+2)(x-2)(x-2) \quad \textit{i.e.,}\ (x-2)(x^2-4)$$

Example 2. Find the L.C.M. of $x^3-19x+30$ and $x^3-10x^2+31x-30$.

Solution. First we factorise given polynomials.

Let $f(x)=x^3-19x+30$ Then $f(2)=8-38+30=0$

So, $x-2$ is a factor of $f(x)$.

$$\begin{aligned} f(x)&=x^3-19x+30\\ &=x^3-2x^2+2x^2-4x-15x+30\\ &=x^2(x-2)+2x(x-2)-15(x-2)\\ &=(x-2)(x^2+2x-15)\\ &=(x-2)[x^2+5x-3x-15]\\ &=(x-2)[x(x+5)-3(x+5)]\\ &=(x-2)(x+5)(x-3) \end{aligned}$$

Let $g(x)=x^3-10x^2+31x-30$ Then $f(2)=8-40+62-30=0$

So, $x-2$ is a factor of $g(x)$.

$$\begin{aligned} g(x)&=x^3-10x^2+31x-30\\ &=x^3-2x^2-8x^2+16x+15x-30\\ &=x^2(x-2)-8x(x-2)+15(x-2)\\ &=(x-2)(x^2-8x+15)\\ &=(x-2)[x^2-5x-3x+15]\\ &=(x-2)[x(x-5)-3(x-5)]\\ &=(x-2)(x-5)(x-3) \end{aligned}$$

Hence the L.C.M. $=(x-2)(x-3)(x+5)(x-5)$.

We find the L.C.M. of two non-zero polynomials in the following steps :

(1) *We express each polynomial in the complete factored form i.e., as a product of numerical factor, linear factor and quadratic factors if linear factors are not possible.*

(2) *Then we find the product of the following—*

(*a*) *L.C.M. of the numerical factors in the given polynomials.*

(*b*) *Common factors with highest power of the given polynomials.*

(*c*) *The factors other than common factors, with highest power, present in the given polynomials.*

(3) *The product so obtained is the required L.C.M.*

Sometimes it is difficult to find factors of the two polynomials. Then we find their H.C.F. by division method. Lastly L.C M. is found by the formula given below :

$$\text{L.C.M. of two polynomials}=\frac{\text{Product of the two polynomials}}{\text{H.C.F. of the two polynomials}}$$

Example 3. Find the L.C.M. of x^3-x^2-5x+2 and x^3+4x^2+x-6.

Solution. First we find the H.C.F. of given polynomials by division method.

$$\begin{array}{rl} & \;\;1 \\ x^3-x^2-5x+2 & \big)\;\overline{x^3+4x^2+x-6} \\ & \;\;\underline{x^3-x^2-5x+2} \\ & \;\;+5x^2+6x-8 \end{array}$$

Now $5x^2+6x-8$ is the new divisor and the new dividend will be $5(x^3-x^2-5x+2)$ *i.e.,* $5x^3-5x^2-25x+10$.

$$\begin{array}{rl}
 & \quad x+11 \\
5x^2+6x-8 & \overline{)\ 5x^3-5x^2-25x+10} \\
 & \quad 5x^3+6x^2-\ 8x \\
\cline{2-2}
 & \quad -11x^2-17x+10 \\
 & \quad -5 \\
\cline{2-2}
 & \quad 55x^2+85x-50 \\
 & \quad 55x^2+66x-88 \\
\cline{2-2}
 & \quad 19x+38
\end{array}$$

The remainder is $19x+38$. The new divisor is $(19x+38)\div19$ *i.e.,* $x+2$ and the new dividend is $5x^2+6x-8$.

$$\begin{array}{rl}
 & \quad 5x-4 \\
x+2 & \overline{)\ 5x^2+\ 6x-8} \\
 & \quad 5x^2+10x \\
\cline{2-2}
 & \quad -\ 4x-8 \\
 & \quad -\ 4x-8 \\
\cline{2-2}
 & \quad 0
\end{array}$$

The H.C.F.$=x+2$

So, the L.C.M.$=\dfrac{f(x)\times g(x)}{h(x)}$

$$=\frac{(x^3-x^2-5x+2)(x^3+4x^2+x-6)}{x+2}$$

$$=(x^2-3x+1)(x^3+4x^2+x-6)$$

Or

$$(x^3-x^2-5x+2)(x^2+2x-3).$$

EXERCISE—6(o)

(Section—A)

Find the L.C.M. by factorisation method of the following polynomials :

1. x^2-1 and x^2+2x+1
2. x^2-4 and x^2-x+2
3. x^2-4 and x^3+8
4. x^2+1 and x^4-1
5. $x^2+9x+20$ and $x^2+11x+28$
6. $x^2-10x+24$ and $x^2-11x+30$
7. $2x^2-8$ and x^2-5x+6
8. $2x^2-2x-12$ and $3x^2+15x+18$

(Section—B)

Find the L.C.M. of the following polynomials :

9. x^3+x^2-3x-9 and x^3-5x^2+9x-9
10. $x^3-2x^2-9x+18$ and $x^3-3x^2-10x+24$
11. $x^3-x^2-10x-8$ and $x^3-2x^2-13x-10$
12. $x^3+6x^2+11x+6$ and $x^3-x^2-24x-36$

(Section—C)

Find the H.C.F. and L.C.M. of the following polynomials :

13. $3x^3+2x^2-11x+4$ and $3x^3+14x^2+13x-8$
14. $6x^3+25x^2+16x+7$ and $6x^3-11x^2-8x-5$
15. $2x^4+x^3-9x^2+8x-2$ and $2x^4-7x^3+11x^2-8x+2$
16. The H.C.F. of two polynomials is $x-7$ and their L.C.M. is $x^3-10x^2+11x+70$. One of the polynomials is $x^2-5x-14$. Find the other polynomial.

REVIEW EXERCISE—VI

(Section—A)

1. State for each of the following statements whether it is True or False :

(*a*) $(x-2)$ is a factor of $(2-x)^2$. ☐

(*b*) $(a+3b)$ is a factor of (a^2-9b^2). ☐

(*c*) $(3a+b)$ is a factor of $(6a^2-b^2)$. ☐

(*d*) $(x-2)$ is a factor of (x^3-4x). ☐

2. One of the factors of $9a^2-b^2$ is $(3a-b)$. What is the other factor ?

3. If one factor of a^2-3a+2 is $(a-1$, find the other factor.

4. Given that $x^2+2x-15=(x+5)(x+k)$, find the value of k.

5. Fill in the blanks to make each of the following a true statement :

(*a*) If $f(-1)=0$, then......... is a factor of $f(x)$, where $f(x)$ is a polynomial in x.

(*b*) $(x+2)$ is .. ... of $2x^3+4x^2-5x-10$ [*C.B.S.E., 1987 (A.I.)*]

(*c*) The H.C.F. of the polynomials $(x-2)(x+3)(x-4)$ and $(x-2)(x+2)(x-5)$ is......... [*C.B.S.E., 1987 (Delhi)*]

(*d*) The H.C.F. and L.C.M. of two polynomials are $(x+1)$ and $2(x+1)(x^2-4)$ respectively. If one polynomial is $(x+1)(x-2)$, then the other is......... [*C.B.S.E., 1987 (A.I.)*]

(Section—B)

Factorise :

6. $2x^2+yz-2xy-xz$.

7. $9a^2-b^2+3a-b$.

8. $x(x+z)-y(y+z)$.

9. $(a+b)^3-(a+b)$.

10. $9-x^2+2xy-y^2$.

11. $32-2(x-4)^2$.

12. $ab(x^2+y^2)+xy(a^2+b^2)$.

13. x^5+x^4-x-1.

14. $a^2+ac+bc-b^2$.

15. $x^2+23x-108$.

16. $x^2-29xy+120y^2$.

17. $x^2-39x-550$.

18. $30x^2-xy-42y^2$.

19: $(4-x)^2-2x$.

20. $(3a-1)^2-6a+2$.

21. $8a^3-27b^3$.

(Section—C)

Factorise the following :

22. x^6-729.

23. x^6-26x^3-27.

24. $x^{12}-y^{12}$. [*C.B.S.E., 1986 (Delhi)*]

25. Find the H.C.F. of $4x^3+8x^2-3x-9$ and $12x^3+28x^2+13x-3$. [*C.B.S.E., 1986 (A.I.)*]

26. Find the L.C.M. of x^3+x^2+x+1 and x^3+2x^2+x+2. [*C.B S.E., 1987 (A.I.)*]

27. The H.C.F. of two polynomials is $x+3$ and their L.C.M. is x^3-7x+6. If one polynomial is x^2+2x-3, find the other. [*C.B.S.E., 1986 (Delhi)*]

☐ ☐

7

LINEAR EQUATIONS IN ONE VARIABLE

7.1. LINEAR EQUATIONS

Any open sentence using the sign '=' is called an **equation**.

Any open sentence using the sign of inequality '>' or '<' is called an **inequation**.

Examine the following equations :

(*a*) $2x+5=7$ (*b*) $3x-7=2x+3$

(*c*) $x^2-9=16$ (*d*) $x^2+7x+12=0$

(*e*) $x^3-7=20$ (*f*) $x^3+4x^2=3x+2$

Observe that equations (*a*) and (*b*) contain only *linear polynomials.* In equations (*c*) and (*d*), only *quadratic polynomials* are involved, whereas in (*e*) and (*f*) *cubic polynomials* are involved.

An equation involving only linear polynomials is called a linear equation.

A linear equation in one variable is of the form $ax+b=0$, where a and b are real numbers and $a \neq 0$.

The number 'a' is called the *coefficient* of x and 'b' is called the *constant term* of the equation.

Let us consider the linear equation

$$3x+5=14$$

We will substitute only natural numbers for x.

Substituting 2 for x, we get

Left Hand Side $=3\times2+5=11$

Right Hand Side $=14$

Therefore, L.H.S.$\neq$R.H.S.

Substituting 3 for x, we get

L.H.S. $=3\times3+5=14$

R.H.S. $=14$

Therefore, L.H.S.$=$R.H.S., when $x=3$.

Here natural numbers are **domain** of the variable x and 3 is the **solution.**

The set of numbers from which the values of the variables in an equation are selected is called the **replacement set** or the **domain** of the variable.

The set of values of the variable which make the equation true is called the **solution set** or the **truth set**.

Example. Examine whether 4 is a solution of

$$5x-3=3x+7$$

Solution. Substituting $x=4$, we get

$\text{L.H.S.}=5\times 4-3=20-3$ *i.e.*, 17

and $\text{R.H.S.}=3\times 4+7=12+7$ *i.e.*, 19

So, $\text{L.H.S.}\neq\text{R.H.S.}$ Therefore 4 is not a solution of the given linear equation.

Solving an equation means finding the truth set of the equation.

The process of solving an equation consists of getting a series of *equivalent* equations such that each succeeding equation is simpler than the preceding one and the last equation directly gives the truth set.

The following properties of equality are used in solving equations :

(*i*) **Addition Property** : An equality remains unchanged when the *same* number is added to both sides of it.

If $a=b$, then $a+c=b+c$.

(*ii*) **Multiplication Property** : An equality remains unchanged when both of its sides are multiplied by the *same* number.

If $a=b$, then $c\times a=c\times b$.

(*iii*) **Cancellation Law for Addition** : An equality remains unchanged when the *same* real number occurring as an addend on both of its sides is cancelled.

If $a+c=b+c$, then $a=b$.

(*iv*) **Cancellation Law for Multiplication** : An equality remain unchanged when both of its sides are divided by the *same* non-zero real number.

If $ac=bc$, then $a=b$.

Example 1. Solve $5x+7=2x+25$, $x\in N$.

Solution. $5x+7=2x+25$

or $5x+7=2x+18+7$

Cancelling same addend 7 from both sides,

$$5x=2x+18$$

Adding $-2x$ to both sides, we get

$$5x-2x=2x+18-2x$$

or $3x=18$

or $3.x=3.6$

Dividing both sides by 3, we get

$$x=6$$

The solution set is $\{6\}$. $(6\in N)$

Example 2. Solve $5x-3=2x+7$, $x\in Q$.

Solution. $5x-3=2x+7$

Adding 3 to both sides, we get

$$5x-3+3=2x+7+3$$

or $5x=2x+10$

Adding $-2x$ to both sides, we get

$$5x-2x=2x+10-2x$$

or $3x=10$

Dividing both sides by 3, we wet

$$\frac{3x}{3}=\frac{10}{3}$$

or $x=\frac{10}{3}$ $\qquad \frac{10}{3}$ is a rational number.

So, the truth set $=\left\{\frac{10}{3}\right\}$.

In the above working, by adding 3 to both sides -3 disappears from the L.H.S. and appears on the R.H.S. with its *sign changed.*

In actual practice, a term may be transferred from one side of an equation to the other, but its sign must be *changed.*

This process is called *transposition.*

Example 3. Solve $7x+3=3x+\frac{5}{7}$, where $x\in R$.

Solution. $7x+3=3x+\frac{5}{7}$.

Bringing variable terms to left hand side and those not involving variable to right hand side, we get

$7x-3x=\frac{5}{7}-3$ [By transposition, changing the sign of $3x$ and 3.]

or $(7-3)x=\frac{5-21}{7}$

or $4x=-\frac{16}{7}$.

∴ $x=-\frac{16}{7\times 4}$

i.e., $x=-\frac{4}{7}$ $\qquad -\frac{4}{7}$ is a real number.

So, the solution set is $\left\{-\frac{4}{7}\right\}$.

EXERCISE—7(a)

Find the solution set for the following equations, taking $x\in N$.

1. $6x+7=2x+31$.
2. $3x+4x+2=37$.

Find the solution set of the following equations, domain being the set of integers :

3. $8x+4=5x+19$.
4. $7x+2=2x-23$.
5. $4(x+5)=x-7$.
6. $28x+1=15$.

Find the solution set for each of the following equations, the replacement set being the set of rational number :

7. $2x+\frac{1}{2}=x+\frac{1}{3}$.
8. $3x+10=x-7$.
9. $7x+10-3x=5$.
10. $8x-5=5x-8$.

Solve the following equations, when $x\in R$.

11. $2x+3=16(2x-3)$.
12. $-5x+2=8x-22-4x$.

Example. Solve $\frac{7x-1}{4}-\frac{1}{3}\left(2x-\frac{1-x}{2}\right)=6\frac{1}{3}$, when $x \in R$.

Solution. $\frac{7x-1}{4}-\frac{1}{3}\left(2x-\frac{1-x}{2}\right)=\frac{19}{3}$

or $\frac{7x-1}{4}-\frac{2x}{3}+\frac{1-x}{6}=\frac{19}{3}$

or $3(7x-1)-4(2x)+2(1-x)=4\times 19$

or $21x-3-8x+2-2x=76$

or $(21-8-2)x-1=76$

or $11x-1=76$

or $11x-1+1=76+1$

or $11x=77$

or $\frac{11x}{11}=\frac{77}{11}$

$\therefore$ $x=7$ and $7 \in R$

The solution set $=\{7\}$

EXERCISE—7 (b)

Solve the following equations. The domain of the variable in each case is R.

(Section—A)

1. $\frac{2}{3}-\frac{x-3}{6}=0.$
2. $4-\frac{x-3}{2}=3.$
3. $(5x-4)-(3x-1)=3$
4. $5x-3(x-2)=3x-2(x-1).$
5. $\frac{x}{3}-\frac{2x-5}{2}=0.$
6. $\frac{4}{3}x-\frac{3}{4}=\frac{2}{3}x-\frac{1}{4}.$
7. $\frac{x-1}{3}-\frac{x-2}{4}=1.$
8. $\sqrt{3}x+5=7.$

(Section—B)

9. $\frac{1}{5}(3x-4)+\frac{1}{6}(3x+1)=\frac{7}{15}.$
10. $\frac{2-3x}{3}+\frac{1+5x}{5}=\frac{3-8x}{4}.$
11. $\frac{9x+7}{2}-\left(x-\frac{x-2}{7}\right)=36.$
12. $(x-2)(x+3)=x^2-4.$
13. $\frac{1}{3}(x-3)-x+3=2(x-1).$
14. $\frac{2}{3}x-\frac{3x+2}{2}-\frac{x+3}{6}+\frac{5}{2}=0.$

(Section—C)

15. $3x-4-3\sqrt{2}=2x+\sqrt{2}.$
16. $5x-3=2x-8+\sqrt{2}x.$
17. $6x+12=\sqrt{5}x+21.$
18. $\sqrt{7}x-\sqrt{3}=\sqrt{3}x+7.$

7.2. APPLICATIONS

Equations can be translated into story problems and *vice versa.*

Study the following example.

Six times a number diminished by 7 comes to 47. What is that number ?

We can translate the given situation into an equation.

Let x be the number.

Six times of the number is $6 \times x$ *i.e.*, $6x$.

Then this number diminished by 7 is $6x-7$.

But this is equal to 47.

$\therefore \quad 6x-7=47$

or $\quad 6x-7+7=47+7$

or $\quad 6x=54$

or $\quad \frac{6x}{6}=\frac{54}{6}$

$\therefore \quad x=9.$ The required number is 9.

Check. $6 \times 9=54$. $54-7=47$.

This agrees with what is given in the problem.

Example 1. Naresh Kumar is three times as old as his son. In 12 years he will be twice as old as his son. Find their ages.

Solution. Let us say that son's present age is x years.

Then Naresh Kumar's age$=3 \times x$ years$=3x$ years.

After 12 years, son's age will be $(x+12)$ years.

After 12 years, Naresh Kumar's age will be $(3x+12)$ years.

According to the given condition,

$3x+12=2(x+12)$

or $\quad 3x+12=2x+24$

or $\quad 3x+12=2x+12+12$

or $\quad 3x=2x+12$

or $\quad 2x+x=2x+12$

$\therefore \quad x=12.$

The age of the son$=12$ years.

The age of Naresh Kumar$=3 \times 12$ years$=36$ years.

Check. After 12 years, $12+12=24$ years

$36+12=48$ years

And $48=2 \times 24$.

This agrees with what is given. Hence our solution is correct

While solving story problem, follow these steps :

(*i*) *Represent the unknown by a letter x (variable).*

(*ii*) *Translate the conditions of the problem into mathematical phrases.*

(*iii*) *Form an equation according to the relationship of conditions.*

(*iv*) *Solve the equation to find the value of x.*

(*v*) *Check the value/values with the help of original problem.*

EXERCISE—7(c)

(Section—A)

1. Five times a number diminished by 8 comes to 27. Find the number.
2. The sum of two consecutive whole numbers is 71. Find the numbers.
3. Three times a number less the number itself gives 20. Find the number.

4. What is the number which when multiplied by 3 is 32 more than itself ?

5. The age of a father is four times the age of his son. The sum of their ages is 85 years. Find their ages.

6. Find three consecutive even numbers whose sum is 84.

7. Find three consecutive odd numbers whose sum is 117.

8. The difference between 12 times the length and 10 times the length of a line is 120 cm. Take the length of the line to be x, and by forming an equation in x, find the length of the line in centimetres.

9. Find a number such that one-fifth of it is less than one-fourth of it by 3.

(Section—B)

10. Find a number such that if 19 be added to it, the sum is 12 less than double the number.

11. Jagdish's age is five times his son's age and 4 years ago the sum of their ages was 28 years. Find their present ages.

12. A father is twice as old as his son. Twenty-five years ago he was seven times as old as his son. What are their ages now ?

13. A number consists of two digits whose sum is 9. If 45 be added to the number, the digits are reversed. Find the number.

14. A number consists of two digits. The sum of its digits is 1 On reversing the digits of the number, the number decreases by 36. Find the number.

15. The sum of two digits of a number is 15. If 9 is added to the number, the digits are reversed. Find the number.

16. The ten's digit of a two-digit number exceeds the unit's digit by 7. The sum of the digit is $\frac{1}{9}$ of the number. Find the number.

17. The numerator of a fraction is 2 less than the denominator. If 21 is added to the numerator, the value of the fraction is doubled Find the original fraction.

(Section—C)

18. A man covers a distance of 15 km in 3 hours, partly by walking and partly by running. If he walks at 3 km/hour and runs at 9 km/hour, find the distance he covers by running.

19. Five shirts and seven half-pants together cost Rs. 232. The cost of a shirt is Rs. 8 more than the cost of a half-pant. Find the cost of each.

20. The present age of a father is 3 years more than three times the age of the son. Three years hence, father's age will be 10 years more than twice the age of the son. Find their present ages. *[C.B.S.E., 1982 (Delhi)]*

21. A bag contains 10 paise coins and 50 paise coins. The number of 10 paise coins is four times the number of 50 paise coins. If the total value is Rs. 9, how many coins are there in each variety ?

22. At a fair, a boy tries his skill in shooting. He was to receive 20 paise for hitting the "bull's eye" and had to pay 8 paise for missing the "bull's eye". He tried, 50 shots but received only 20 paise. How many times did he hit the "bull's eye" ?

Example 1. Subir bought two radio sets for Rs 1200. He sold one at a loss of 14% and the other at a profit of 14%. If the selling price of each set is the same, find the cost of each.

Solution. Let the cost of first set be Rs x.

Then the cost of second set is Rs $(1200-x)$.

For first radio set, C.P.=Rs x and loss=14%

$\therefore$ S.P. of the first set=Rs $\frac{86\times x}{100}$.

For second radio set, C.P.=Rs $(1200-x)$ and gain=14%

$\therefore$ S.P. of the second set=Rs. $\frac{114\times(1200-x)}{100}$.

But S.P. of both sets is the same.

$$\therefore \quad \frac{86\times x}{100}=\frac{114\times(1200-x)}{100}$$

or $\quad 86\times x=114\times(1200-x)$

or $\quad 43x=57\times(1200-x)$

or $\quad 43x=57\times1200-57x$

or $\quad 43x+57x=57\times1200$

or $\quad 100x=57\times1200$

or $\quad x=57\times12$

i.e., $\quad x=684.$

So, C.P. of the first set $=$Rs 684

C.P. of the second set $=$Rs 1200$-$Rs 684$=$Rs 516.

Example 2. A man rowing at the rate of 5 km an hour in still water takes thrice as much time in going 40 km up the river as in going 40 km down. Find the rate at which the river flows.

[*C.B S.E., 1982 (A.I.)*]

Solution. Let the rate at which river flows be x km per hour.

The man rows at the rate of 5 km/hour in still water.

$\therefore$ Speed of the man downstream$=(5+x)$ km/hour.

Also speed of the man upstream$=(5-x)$ km/hour.

Then time taken in rowing 40 km downstream$=\frac{40}{5+x}$ hours.

And time taken in rowing 40 km upstream$=\frac{40}{5-x}$ hours.

According to the given condition,

$$\frac{40}{5-x}=3\left(\frac{40}{5+x}\right)$$

or $\quad 40\,(5+x)=3\times40\,(5-x)$

or $\quad 5+x=3\,(5-x)$

or $\quad 5+x=15-3x$

or $\quad x+3x=15-5$

or $\quad 4x=10$

$\therefore \quad x=2{\cdot}5$

ie.., the rate at which the river flows is 2·5 km/hour.

EXERCISE—7 (d)

(Section—A)

1. The perimeter of a rectangle is 350 metres. Its sides are in the ratio 5 : 2. Find the area of the rectangle.
2. The width of a room is three-fifths of its length. If the length had been 4 metres less and width 4 metres more, the room would have been a square. Find its dimensions.
3. Two trains with speeds 32 km and 24 km per hour respectively, starting at the same time, run in opposite directions between two stations *A* and *B*, 336 km apart, one going from *P* to *Q* and the other from *Q* to *P*. Find when and where they will meet ?
4. Mohan Singh had Rs 2000. He lent part of it at 10% simple interest and the rest at 8%. The total annual interest received was Rs 184. How much did he lend at 10% ?
5. A certain number of men do a piece of work in 40 days. If there had been 45 men more, the work could have been finished in 25 days. How many men were employed to do the work ?

(Section—B)

6. The length of a lawn is thrice its breadth. If length is increased by 4 metres and breadth is diminished by 1 metre, the area does not change. Find its dimensions.
7. The length of a hall exceeds its breadth by 6 metres. If the length be increased by 4 metres and the breadth diminished by 2 metres, the area is increased by 4 sq. metres. Find the dimensions of the room.
8. In a co-educational school there are 30 more girls than boys. If the number of girls was increased by 10% and the number of boys by 16%, there would be 18 more girls. Find the number of pupils in the school.
9. At noon Sohan starts to ride at 8 km an hour ; two hours later Ramu starts after him on a bicycle at 12 km an hour. How far will Sohan have ridden before he is overtaken by Ramu ?

   Find also at what time Sohan and Ramu will be 8 km apart.
10. The population of a town is 15,000 ; the annual death-rate of males is 4½% and that of females is 5½%. The total number of deaths during the year is 745. Find the number of males and females in the village.
11. A path 1·5 metres in width is laid all round outside a square lawn and covers an area of 129 sq. metres. Find the area of the lawn excluding the path.

(Section—C)

12. Two partners Rahim and Karim together lend Rs 3364 at 5% compounded annually. The amount Rahim get at the end of 3 years is the same as Karim gets at the end of 5 years. Find the share of each in the principal.
13. Karim bought a tonga and a horse for Rs. 6000. He sold the tonga at a gain of 10% and the horse at a gain of 15%, making a profit of 12% on the whole. Find the cost of the tonga and the horse.
14. *A* and *B* can do a piece of work in 9 days. A alone can do it in 12 days. In how many days can *B* alone do it ?
15. Two cyclists *A* and *B* start from the same place to ride in the same direction. A starts at noon at 7·5 km and *B* starts at 2 P.M. at 10 km an hour.

    (*a*) How far will *A* have gone before he is overtaken by *B* ?

    (*b*) At what times are *A* and *B* five kilometres apart ?
16. The speed of a boat in still water is 8 km per hour. It goes 20 km down stream. If it can go 12 km upstream in the same time, find the speed of the stream.

REVIEW EXERCISE—VII

(Section—A)

1. If $x+5 : 2 = 18 : 3$, find x.
2. Solve : $\dfrac{5(x+3)}{x} = 8$.
3. Examine whether $x = -1$ is a solution of $24 - 3(x-2) = x+8$.
4. Solve the equation $\dfrac{3x-1}{4\frac{1}{2}} - \dfrac{x-1}{2} = 0$ and verify the solution.
5. Solve : $\dfrac{x-2}{4} + \dfrac{1}{3} = x - \dfrac{2x-1}{3}$.
6. Solve : $\dfrac{2}{3}x - \dfrac{3x+2}{2} - \dfrac{x+3}{6} + \dfrac{5}{2} = 0,\ x \in R$. [*C.B.S.E., 1979 (Delhi)*]
7. Solve : $\dfrac{5x}{3} - \dfrac{x-2}{3} = 2\dfrac{1}{3} - \dfrac{1}{2}\left(x - \dfrac{2x-1}{3}\right),\ x \in R$.

(Section—B)

8. A square has the same area as a rectangle whose length is 5 cm longer, and breadth 4 cm shorter than the side of the square. Find the length and breadth of the rectangle.
9. Divide Rs. 3100 into two parts such that if one part be put out 5% per annum and the other at 8% per annum, the total yearly income may be Rs. 200.
10. Solve : $\dfrac{1}{x-1} + \dfrac{2}{x-2} = \dfrac{3}{x-3}$. [*C.B.S.E., 1982 (A.I.)*]
11. Solve : $\dfrac{1}{x+1} + \dfrac{1}{x+5} = \dfrac{1}{x+2} + \dfrac{1}{x+4}$. [*C.B.S.E, 1983 (A.I.)*]
12. The difference of ages Sohrab and his father is 30 years. The difference of the squares of their ages is 1560. Find the ages of the son and his father. [*C.B.S.E., 1983 (Delhi)*]

(Section—C)

13. How many grams of pure gold be added to 150 grams of alloy 70% pure to make the alloy 85% pure ?
14. Divide Rs. 12,615 between Ram and Shyam so that the amount that Raw receives in 2 years and Shyam in 4 years are equal. Interest is 5% per annum, compounded annually. [*C.B.S.E., 1984 (Delhi)*]

□ □

8

LOGARITHMS

8.1. INDICES

If m is a positive integer, then

$a \times a \times a \times a \times$ ....upto m factors$=a^m$. a^m is read as 'mth power of a'. The repeated factor 'a' is called the **base**, and 'm' is called the **power** or **index** or **exponent.**

Example. $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$.

You are already familiar with the following laws of indices :

(*i*) $a^m \times a^n = a^{m+n}$,

(*ii*) $a^m \div a^n = a^{m-n}$, $m > n$,

(*iii*) $(a^m)^n = (a^{mn})$,

(*iv*) $(ab)^m = a^m b^m$,

(*v*) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$, $b \neq 0$,

when m and n are positive integers ; a and b are real numbers.

Example 1. Simplify $x^{2n} \cdot x^{n-3} \cdot x^5$.

Solution.
$$x^{2n} \cdot x^{n-3} \cdot x^5 = x^{2n+n-3+5}.$$
$$= x^{3n+2}$$

Example. Simplify $35x^7 y^4 z^2 \div 7x^3 y^2 z$.

Solution.
$$35x^7 \ y^4 \ z^2 \div 7x^3 \ y^2 \ z = \frac{35x^7 \ y^4 \ z^2}{7x^3 \ y^2 \ z}$$
$$= 5x^{7-3} \ y^{4-2} \ z^{2-1}$$
$$= 5x^4 \ y^2 \ z.$$

Example 3. Simplify $\left(\frac{a^3 \ b^5}{a \ b^2}\right)^4$

Solution.
$$\left(\frac{a^3 \ b^5}{a \ b^2}\right)^4 = (a^{3-1} \ b^{5-2})^4$$
$$= (a^2 \ b^3)^4$$
$$= (a^2)^4 \ (b^3)^4$$
$$= a^8 \ b^{12}.$$

EXERCISE—8 (a)

(Section—A)

Simplify :

1. $a^5 \times a^3 \times a \times a^2$.
2. $(x^2)^5 \times (x^3)^2$.
3. $(6x^7 \div 3x^4)^3$.
4. $\left(\frac{2x^2}{5y^3}\right)^4$.
5. $x^4 \times x^7 \times x^5 \div x^{15}$.
6. $a^5 \div a^7 \div a^9 \times a^8$.
7. $x^{2a+b-c} \,.\, x^{2c+a-b} \,.\, x^{2b+c-a}$.
8. $(x^a)^{b-c} \times (x^b)^{c-a} \times (x^c)^{a+b}$.

(Section—B)

Simplify :

9. $\dfrac{x^{a+b} \times x^{b+c} \times x^{c+a}}{(x^a \times x^b \times x^c)^2}$
10. $(a^3\, b^2)^m \times (ab^4)^u$.
11. $(a^2b^3)^m \div (a^5\, b^4)^n$.
12. $(xyz^2)^m \times (xy^2z^3)^n \times (x^2\, y^3\, z^5)^p$.
13. $[(x^{a+b+c} \times x^{a-b+c})^b]^c$.
14. $[(x^{2a-b+3c} \div x^{a-2b+2c})^c]^a$.
15. $(x^{b+c})^a \times (x^{c-a})^b \div (x^{a-b})^c$.
16. $\left(\frac{x^a}{x^b}\right)^{a+b} \times \left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^a}{x^c}\right)^{a+c}$

(Section—C)

Simplify :

17. $\left(\frac{x^{a+b}}{x^{2b}}\right) \times \left(\frac{x^{b+c}}{x^{2c}}\right) \times \left(\frac{x^{c+a}}{x^a}\right)$.
18. $\left(\frac{x^{a^2+b^2}}{x^{ab}}\right)^{a+b} \times \left(\frac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \times \left(\frac{x^{c^2+a^2}}{x^{ca}}\right)^{c+a}$.
19. $\left(\frac{x^a}{x^b}\right)^{a^2+ab^2+b^2} \times \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \div \left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$

8.2. RATIONAL INDICES

The five laws of indices have been deduced by assuming that m and n are positive integers. Then $a^{1/m}$, a^{-m}, a^0, $a^{m/n}$, etc., have no meaning.

By assuming $a^m \times a^n = a^{m+n}$ true for all values of m and n, positive or negative integral or fractional, we can deduce meanings for $a^{m/n}$, a^0, a^{-n} and $a^{1/n}$.

(*i*) *Any non-zero number raised to zero power is equal to 1.*

$a^0 = 1$, where a is any real number other than 0.

If a is zero, then a^0 is *not* defined.

Examples. $5^0 = 1$, $\left(\frac{1}{3}\right)^0 = 1$, $(a^2b^5)^0 = 1$.

(*ii*) *Any non-zero number raised to a negative power is multiplicative inverse of the same number raised to the corresponding positive power.*

$$a^{-m} = a^{1/m}, \text{ where } a \neq 0.$$

Examples. $3^{-2} = \frac{1}{3^2}$, $x^{-5} = \frac{1}{x^5}$, $(xy)^{-3} = \frac{1}{(xy)^3}$

(*iii*) *If a is a positive real number, m an integer and n a positive integer then,*

$$a^{m/n} = \sqrt[n]{a^m} \quad OR \quad a^{m/n} = (\sqrt[n]{a})^m$$

Thus, $a^{m/n}$ may be interpreted either as the nth root of mth power of a or as the mth power of the nth root of a.

In particular, $a^{1/2}=\sqrt{a}$, $a^{1/3}=\sqrt[3]{a}$, $a^{1/4}=\sqrt[4]{a}$, $a^{1/n}=\sqrt[n]{a}$.

Examples. $2^{1/2}=\sqrt{2}$, $4^{1/3}=\sqrt[3]{4}$, $(25)^{1/4}=\sqrt[4]{25}$.

With the help of these meanings, the remaining laws of indices can be proved for all integers, positive or negative and fractional numbers.

Since laws of indices are universally true, all the operations of multiplication, division, involution and evolution are also applicable to expressions containing fractional and negative indices.

Example 1. Find the value $(1024)^{-4/5}$.

Solution.
$$\begin{aligned}(1024)^{-4/5}&=(2^{10})^{-4/5}=2^{10-(4/5)}\\&=2^{-8}\\&=\frac{1}{2^8}\qquad=\frac{1}{256}\end{aligned}$$

Example 2. Simplify $(a^{-1/2}\,b^{3/4})^{-2/3}\div(a^{4/9}\,b^{-1/3})^{3/2}$

Solution.
$$\begin{aligned}(a^{-1/2}\,b^{3/4})^{-2/3}\div(a^{4/9}\,b^{-1/3})^{3/2}&=\frac{(a^{-1/2}\,b^{3/4})^{-2/3}}{(a^{4/9}\,b^{-1/3})^{3/2}}\\&=(a^{-1/2}\,b^{3/4})^{-2/3}\times(a^{4/9}\,b^{-1/3})^{-3/2}\\&=(a^{-1/2})^{-2/3}\,(b^{3/4})^{-2/3}\times(a^{4/9})^{-3/2}\,(b^{-1/3})^{-3/2}\\&=a^{1/3}\,.\,b^{-1/2}\,.\,a^{-2/3}\,.\,b^{1/2}\\&=a^{1/3-2/3}\,.\,b^{-1/2+1/2}\\&=a^{-1/3}\,.\,b^{0}\\&=\frac{1}{a^{1/3}}\,.\,1=\frac{1}{\sqrt[3]{a}}\end{aligned}$$

Example 3. Simplify
$$\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2}\left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2}\left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}$$

Solution. The given expression
$$\begin{aligned}&=\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2}\,.\left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2}\left(\frac{x^c}{x^a}\right)^{c^2+ca+a^2}\\&=\left(x^{a-b}\right)^{a^2+ab+b^2}\,.\,\left(x^{b-c}\right)^{b^2+bc+c^2}\,.\,\left(x^{c-a}\right)^{c^2+ca+a^2}\\&=x^{(a-b)\,(a^2+ab+b^2)}\,.\,x^{(b-c)\,(b^2+bc+c^2)}\,.\,x^{(c-a)\,(c^2+ca+a^2)}\\&=x^{a^3-b^3}\,.\,x^{b^3-c^3}\,.\,x^{c^3-a^3}\\&=x^{a^3-b^3+b^3-c^3+c^3-a^3}\\&=x^0\\&=1.\end{aligned}$$

EXERCISE—8 (b)

(Section—A)

1. Find the value of :

(a) $(625)^{-1/4}$. (b) $(27)^{4/3}$. (c) $9^{-3/2}$.

(d) $(128)^{-3/7}$. (e) $\dfrac{1}{\sqrt{16^{3/2}}}$. (f) $\left(\dfrac{27}{64}\right)^{-2/3}$.

2. Simplify : $5^0 \times 4^{-} + 8^{1/3}$
3. Simplify and express with positive exponents :
   (*a*) $(3x^2)^0 \cdot (xy)^{-2}$ (*b*) $(-27a^9)^{2/3}$

(Section—B)

4. Simplify :
   (*a*) $(27b^{-6}c^3)^{1/3}$. (*b*) $(8^{-2} \times 9^{3/4})^{-2/3}$. (*c*) $(\sqrt[8]{a^{12}b^{-6}})^{-4}$
   (*d*) $\sqrt{x^{-6}y^8} \times \sqrt[4]{x^4y^{-12}}$.
5. Find the value of $\sqrt[3]{(16)^{-3/4}(125)^{-2}}$.
6. Simplify : $\left(\dfrac{x^l}{x^m}\right)^{l+m} \times \left(\dfrac{x^m}{x^n}\right)^{m+n} \times \left(\dfrac{x^n}{x^l}\right)^{n+l}$.
7. Simplify : $\dfrac{2^n+2^{n-1}}{2^{n+1}-2^n}$
8. Simplify : $\dfrac{3^{n+4}-6\times 3^{n+1}}{7\times 3^{n+2}}$.
9. Simplify : $\dfrac{a+b+c}{a^{-1}b^{-1}+b^{-1}c^{-1}+c^{-1}a^{-1}}$.
10. Simplify : $\sqrt[ab]{\dfrac{x^a}{x^b}} \times \sqrt[bc]{\dfrac{x^b}{x^c}} \times \sqrt[ca]{\dfrac{x^c}{x^a}}$.
11. Show that $\dfrac{y^{-1}}{x^{-}+y^{-1}} + \dfrac{y^{-1}}{x^{-1}-y^{-1}} = \dfrac{2xy}{y^2-x^2}$.

(Section—C)

12. Prove that

$$\frac{\left(p+\dfrac{1}{q}\right)^m \times \left(p-\dfrac{1}{q}\right)^n}{\left(q+\dfrac{1}{p}\right)^m \times \left(q-\dfrac{1}{p}\right)^n} = \left(\frac{p}{q}\right)^{m+n}.$$

13. Show that $\left(\dfrac{x^m}{x^n}\right)^{m+n-l} \times \left(\dfrac{x^n}{x^l}\right)^{n+l-m} \times \left(\dfrac{x^l}{x^m}\right)^{l+m-n} = 1$.
14. Simplify : $\left(\dfrac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \times \left(\dfrac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \times \left(\dfrac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$
15. Show that $\dfrac{1}{1+a^{y-x}+a^{z-x}} + \dfrac{1}{1+a^{x-y}+a^{z-v}} + \dfrac{1}{1+a^{y-z}+a^{x-z}} = 1$.
16. Simplify : $\left(\dfrac{x^{ab}}{x^{a^2+b^2}}\right)^{a+b} \times \left(\dfrac{x^{b^2+c^2}}{x^{bc}}\right)^{b+c} \times \left(\dfrac{x^{ca}}{x^{c^2+a^2}}\right)^{c+a}$.

8.3. MEANING OF LOGARITHMS

The word '**logarithm**' has been derived from the Greek words *Logos* and *Arithmos*. Logos means ratio and Arithmos means number. It was discovered by a Scottish mathematician **John Napier** (1550—1617).

We use logarithms for computations with numbers. The logarithms use the idea of *base* and *power*. In this chapter, we will be finding logarithms of positive real numbers.

$2^3=8$	we say	*'3 is the logarithm of 8 to the base 2'*
	we write	$\log_3 8=3$
$5^3=125$	we say	*'3 is the logarithm of 125 to the base 5'.*
	we write	$\log_5 125=3$
$3^{-2}=\frac{1}{9}$	we say	*'-2 is the logarithm of $\frac{1}{9}$ to the base 3'.*
	we write	$\log_3 \frac{1}{9}=-2.$

The logarithm of any number to a given base is the power to which the base must be raised in order to equal the given number.

If a, x, n are numbers such that $a^x=n$, where $a>1$.

then we say $\log_a n=x$

$\log_a n$ is read as *'logarithm of n to the base a'*.

Thus $\quad 2^5=32 \quad \Leftrightarrow \quad \log_2 32=5$

$$3^{-4}=\frac{1}{81} \quad \Leftrightarrow \quad \log_3 \frac{1}{81}=-4$$

The definition of logarithm helps us to find the logarithms of those numbers which can be expressed as powers of the numbers to the given base or its prime factorisation.

Example 1. Find $\log_7 343$.

Solution. Let $\quad \log_7 343=x \quad \Leftrightarrow \quad (7)^x=143$

$$\Leftrightarrow \quad (7)^x=(7)^3 \quad \Leftrightarrow \quad x=3$$

Hence $\quad \log_7 343=3.$

Example 2. Find the logarithm of 324 to the base $3\sqrt{2}$.

Solution. Let x be the required logarithm.

Then $\quad \log_{3\sqrt{2}} (324)=x \quad \Leftrightarrow \quad (3\sqrt{2})^x=324$

$$\Leftrightarrow \quad (3\sqrt{2})^x=4\times 81 \quad \Leftrightarrow \quad (3\sqrt{2})^x=(\sqrt{2})^4\times(3)^4$$

$$\Leftrightarrow \quad (3\sqrt{2})^x=(3\sqrt{2})^4 \quad \Leftrightarrow \quad x=4$$

Hence $\quad \log_{3\sqrt{2}} 324=4.$

Example 3. If $\log_x 64=-3$, find x.

Solution. Given $\log_x 64=-3 \quad \Leftrightarrow \quad x^{-3}=64$

$$\Leftrightarrow \quad x^{-3}=(4^{-1})^{-3}$$

$$\Leftrightarrow \quad x=4^{-1}=\frac{1}{4}.$$

EXERCISE—8 (c)

(Section—A)

1. Change the following exponential forms to logarithmic forms :

(*a*) $2^7=128$ (*b*) $8^{-2/3}=\frac{1}{4}$ (*c*) $[5\sqrt{5}]^{8/3}=625$

2. Express the following in exponential form :

(*a*) $\log_6 216=3$ (*b*) $6=\log_{\sqrt{5}} 125$ (*c*) $-3=\log_5 8.$

(Section—B)

3. Evaluate the following :

(a) $\log_4 256$ (b) $\log_9 \left(\frac{1}{27}\right)$ (c) $\log_{49} 343\sqrt{7}$.

4. Find the logarithm of

(a) 144 to the base $2\sqrt{3}$ (b) $\sqrt{5}$ to the base ·008.

5. Find the value of x, if

(a) $\log_x 4=\frac{2}{3}$ (b) $\log_x {\cdot}0625=-2$.

8.4. LAWS OF LOGARITHMS

Examine the following cases :

$3^4=81 \Leftrightarrow \log_3 81=4$; $9^2=81 \Leftrightarrow \log_9 81=2$.

We notice that **logarithm of the same number is different for different base.**

Consider the following examples :

$(5)^0=1$; $\left(\frac{3}{4}\right)^0=1$; $x^0=1$ when $x\neq 0$.

For all *non-zero* values of a, we know that $a^0=1$

Then $\log_a 1=0$

Thus **logarithm of 1 to any base is zero.**

Again for all values of a, we know that

$a^1=a$ Then $\log_a a=1$

Thus **logarithm of a number to itself as base is 1.**

(1) The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers to the same base.

i.e., $\forall\ m, n, a>0$, $\log_a (mn)=\log_a m+\log_a n$

Example. $\log_{10} 35=\log_{10} (5\times 7)$

$=\log_{10} 5+\log_{10} 7$

(2) The logarithm of a quotient of two numbers is equal to the difference of the logarithms of the numerator and the denominator.

i.e., $\forall\ m, n, a>0$, $\log_a \frac{m}{n}=\log_a m-\log_a n$

Example. $\log_{10} \frac{7}{5}=\log_{10} 7-\log_{10} 5$.

(3) The logarithm of the nth power of a number is n times the logarithm of the number.

i.e., $\forall\ m, a>0$, $\log_a m^n=n \log_a m$. Here $n\in Q$.

Example. $\log_{10} 5^3=3 \log_{10} 5$.

Example 1. Show that $\log_a x^2 (x+1)=2 \log_a x+\log_a (x+1)$.

Solution. $\log_a x^2 (x+1)=\log_a x^2+\log_a (x+1)$

$=2 \log_a x+\log_a (x+1)$

Example 2. Express $\log_a \sqrt{x^m y^n \div \sqrt[3]{z}}$ in terms of $\log_a x$, $\log_a y$ and $\log_a z$

Solution. $\log_a \sqrt{x^m y^n \div \sqrt[3]{z}} = \log_a (x^m y^n \div \sqrt[3]{z})^{1/2}$

$$\begin{aligned}
&= \tfrac{1}{2} \log_a \left(\frac{x^m y^n}{\sqrt[3]{z}} \right) \\
&= \tfrac{1}{2} [\log_a (x^m y^n) - \log_a \sqrt[3]{z}] \\
&= \tfrac{1}{2} (\log_a x^m + \log_a y^n - \log_a z^{1/3}) \\
&= \tfrac{1}{2} (m \log_a x + n \log_a y - \tfrac{1}{3} \log_a z).
\end{aligned}$$

Example 3. If $a^2+b^2=7ab$, prove that

$$\log_p \left(\frac{a+b}{3} \right) = \frac{1}{2} (\log_p a + \log_p b)$$

Solution. Given $a^2+b^2=7ab$

$$\begin{aligned}
\Leftrightarrow \quad & a^2+b^2+2ab=7ab+2ab \\
\Leftrightarrow \quad & (a+b)^2=9ab \\
\Leftrightarrow \quad & a+b=\sqrt{9ab} \\
\Leftrightarrow \quad & a+b=3\sqrt{ab} \\
\Leftrightarrow \quad & \frac{a+b}{3}=\sqrt{ab} \\
\Leftrightarrow \quad & \log_p \left(\frac{a+b}{3} \right) = \log_p \sqrt{ab} \\
& = \log_p (ab)^{1/2} \\
& = \frac{1}{2} \log_p (ab) \\
& = \frac{1}{2} (\log_p a + \log_p b)
\end{aligned}$$

EXERCISE—8(d)

(Section—A)

1. Prove that $\log_a (x^2-1) = \log_a(x+1) + \log_a(x-1)$.
2. Show that $\log_a (1+2+3) = \log_a 1 + \log_a 2 + \log_a 3$.
3. Prove that $(\log_p a)^2 - (\log_p b)^2 = (\log_p ab) \left[\log_p \frac{a}{b} \right]$
4. Prove that $\log_p \frac{x^m}{y^m} + \log_p \frac{y^m}{z^m} + \log_p \frac{}{x^m} \quad 0.$
5. If $\log 2 = x$ and $\log 3 = y$, find
   (*i*) $\log \sqrt{6}$ and (*ii*) $\log 1{\cdot}5$ in terms of x and y.
6. Given $\log_{10} 2 = a$, $\log_{10} 3 = b$, find in terms of a and b,
   (*i*) $\log_{10} 24$ (*ii*) $\log_{10} 3{\cdot}6$.

(Section—B)

Express the following in terms of $\log_a x$, $\log_a y$, $\log^a z$:

7. $\log_a x^2yz^3$.
8. $\log_a x^3y\sqrt{z}$.
9. $\log_a \sqrt{x^2y^3z^4}$.
10. $\log_a \sqrt[3]{x^3y^3 \div z}$.

11. If $a^3+b^3=0$ and $a+b\neq 0$, prove that

$$\log_p (a+b)=\frac{1}{2}\,(\log_p a+\log_p b+\log_p 3).$$

12. If $a^2+b^2=7ab$, prove that

$$2\log_p (a-b)=\log_p 5+\log_p a+\log_p b.$$

13. If $a^2+b^2=6ab$, prove that

$$\log_p \left[\frac{a-b}{2}\right]=\frac{1}{2}\,(\log_p a+\log_p b).$$

(Section—C)

14. Given $\log_{10} x=a$ and $\log_{10} y=b$.
   (*a*) Write down 10^{a-1} in terms of x.
   (*b*) Write down 10^{2b} in terms of y.
   (*c*) If $\log_{10} P=2a-b$, express P in terms of x and y.

15. Show that $7\log_a \frac{16}{15}+5\log_a \frac{25}{24}+3\log_b \frac{81}{80}=\log_b 2$

16. Show that $3\log_b \frac{14}{16}-\log_b \frac{343}{216}+\log_b \frac{64}{9}=\log_c 3.$

8.5. COMMON LOGARITHMS

Since our system of numeration has base ten, we use logarithms to the base ten in all computations with numbers. Logarithms to the base 10 are called **common logarithms**. As we shall work only in base 10, we agree to omit the suffix denoting the base, thus, instead of $\log_{10} 2$, $\log_{10} 15$, etc., we will simply write log 2, log 15, etc.

8.6. CHARACTERISTIC AND MANTISSA

We know that the logarithm of a number to the same base is equal to 1.

Then $\quad \log_{10} 10=1$

$$\log_{10} 100=\log_{10} 10^2=2\log_{10} 10=2\times 1=2$$

Thus the logarithm of any number lying between 10 and 100 lies between 1 and 2 *i.e.*, the values of log 17, log 46, log 90,...... will lie between 1 and 2 and hence they will be of the form (1+*a* fraction).

Therefore, the logarithms of numbers are not always integral, but they are made up of two parts—*an integral part plus a decimal part*.

Again $\log_{10} {\cdot}1=\log_{10}\left(\frac{1}{10}\right)=\log_{10} (10)^{-1}=-\log_{10} 10=-1$

$$\log_{10} {\cdot}01=\log_{10}\left(\frac{1}{100}\right)=\log_{10} (10)^{-2}=-2\log_{10} 10=-2$$

Thus the logarithm of any number lying between ·1 and ·01 lies between −1 and −2. Therefore the logarithms of numbers are not always positive.

The integral part of the logarithm of a number, obtained after expressing the decimal part as *positive*, is called the **characteristic** and the *positive* decimal part is called the **mantissa**

In short, *the decimal part of a logarithm is its mantissa and the number to the left of the decimal is its characteristic.*

Let us determine the characteristic of the logarithm of any number *greater than one.*

$10^0=1 \quad \Leftrightarrow \quad \log_{10} 1=0$ ...(*i*)
$10^1=10 \quad \Leftrightarrow \quad \log_{10} 10=1$ ...(*ii*)
$10^2=100 \quad \Leftrightarrow \quad \log_{10} 100=2$ ...(*iii*)
$10^3=1000 \quad \Leftrightarrow \quad \log_{10} 1000=3$ ...(*iv*)

and so on.

From (*i*) and (*ii*), it follows that the logarithm of any number lying between 1 and 10 (*i.e., a number having one digit in the integral part*) lies between 0 and 1 ; and, therefore, it has only a decimal part. Hence *its characteristic is zero.*

From (*ii*) and (*iii*), it follows that the logarithm of any number lying between 10 and 100 (*i.e., a number having two digits in the integral part*) lies between 1 and 2, and therefore, it has 1 as its integral part. Hence *its characteristic is 1.*

From (*iii*) and (*iv*), it follows that the logarithm of any number lying between 100 and 1000 (*i.e., a number having three digits in the integral part*) lies between 2 and 3, and therefore it has 2 as integral part. Hence *its characteristic is 2.*

Similarly, we can show that the characteristic of the logarithm of a number, having four digits in the integral part is 3.

Thus, **the characteristic of the logarithm of a number greater than 1 is positive and one less than the number of digits in the integral part of the number.**

Let us determine the characteristic of the logarithm of a *decimal fraction.*

$10^0=1 \quad \Leftrightarrow \quad \log_{10} 1=0$ ...(*i*)

$10^{-1}=\frac{1}{10}={\cdot}1 \quad \Leftrightarrow \quad \log_{10} {\cdot}1=-1$ ...(*ii*)

$10^{-2}=\frac{1}{100}={\cdot}01 \quad \Leftrightarrow \quad \log_{10} {\cdot}01=-2$ ...(*iii*)

$10^{-3}=\frac{1}{1000}={\cdot}001 \quad \Leftrightarrow \quad \log_{10} {\cdot}001=-3$ ...(*iv*)

From (*i*) and (*ii*), it follows that the logarithm of any number lying between 1 and ·1 (*i.e., decimal fraction having no zero just after the decimal point*) lies between 0 and −1, and, therefore, it is equal to (−1+*a* positive proper fraction). Hence, *its characteristic is* −1 and is written as $\bar{1}$ and read as "bar one".

From (*ii*) and (*iii*), it follows that the logarithm of any number lying between ·1 and ·01 (*i.e., a decimal fraction having one zero just after the decimal point*) lies between −1 and −2 and therefore, it is equal to (−2+*a* positive proper fraction). Hence, *its characteristic is* $\bar{2}$.

From (*iii*) and (*iv*), it follows that the logarithm of any number lying between ·01 and ·001 (*i.e., a decimal fraction having two zero just after the decimal point*) lies between −2 and −3, and, therefore, it is equal to (−3+*a* positive proper fraction). Hence, *its characteristic is* $\bar{3}$.

Thus, **the characteristic of the logarithm of a number less than 1 is negative and is numerically one more than the number of zeros just after the decimal point.**

The characteristic of the logarithm of a number can be found by *inspection.*

The mantissa of the logarithm of a number is obtained from a table called **logarithmic table.**

8.7. OPERATIONS WITH LOGARITHMS

We follow algebraic rules in performing four fundamental operations on logarithms of numbers. The following examples will illustrate the methods.

Example 1. Add $2{\cdot}3579$, $3{\cdot}2074$ and $\bar{1}{\cdot}7243$.

Solution.

$$\begin{aligned}
&2{\cdot}3579+3{\cdot}2074+\bar{1}{\cdot}7243\\
&=2{\cdot}3579+3{\cdot}2074+(-1+{\cdot}7243)\\
&=5{\cdot}5653-1+0{\cdot}7243\\
&=6{\cdot}2896-1\\
&=5{\cdot}2826.
\end{aligned}$$

Example 2. Add $\bar{2}{\cdot}5931$ and $\bar{1}{\cdot}8547$.

Solution.

$$\begin{aligned}
\bar{2}{\cdot}5931+\bar{1}{\cdot}8547&=(-2+0{\cdot}5931)+(-1+0{\cdot}8547)\\
&=-2-1+0{\cdot}5931+0{\cdot}8547\\
&=-3+1{\cdot}4478\\
&=-3+1+0{\cdot}4478\\
&=-2+0{\cdot}4478\\
&=\bar{2}{\cdot}4478.
\end{aligned}$$

Example 3. Subtract $\bar{2}{\cdot}7285$ from $\bar{3}{\cdot}2594$.

Solution.

$$\begin{aligned}
\bar{3}{\cdot}2594-\bar{2}{\cdot}7285&=(-3+0{\cdot}2594)-(-2+0{\cdot}7285)\\
&=-3+0{\cdot}2594+2-0{\cdot}7285\\
&=-1+0{\cdot}2594-0{\cdot}7285\\
&=-1-0{\cdot}4691\\
&=-1-1+(1-0{\cdot}4691) && \text{[Note this step]}\\
&=-2+0{\cdot}5309\\
&=\bar{2}{\cdot}5309.
\end{aligned}$$

Note that the mantissa is always kept positive.

Example 4. Simplify $\bar{2}{\cdot}3807\times3$.

Solution.

$$\begin{aligned}
\bar{2}{\cdot}3807\times3&=(-2+0{\cdot}3807)\times3\\
&=-6+1{\cdot}1421\\
&=-6+1+0{\cdot}1421\\
&=-5+0{\cdot}1421\\
&=\bar{5}{\cdot}1421.
\end{aligned}$$

Example 5. Simplify $\bar{3}{\cdot}7265\div5$.

Solution.

$$\begin{aligned}
\bar{3}{\cdot}7265\div5&=\frac{\bar{3}{\cdot}7265}{5}\\
&=\tfrac{1}{5}(-3+0{\cdot}7265)\\
&=\tfrac{1}{5}(-3-2+2+0{\cdot}7265) && \text{[Note this step]}\\
&=\tfrac{1}{5}(-5+2{\cdot}7265)\\
&=-1+0{\cdot}5453\\
&=\bar{1}{\cdot}5453.
\end{aligned}$$

Note that by suitable addition and subtraction, the negative characteristic is made divisible by 5.

EXERCISE—8(e)

Simplify :

1. (*a*) $4\cdot7295+2\cdot8537$ (*b*) $3\cdot9457+\bar{1}\cdot5831$

 (*c*) $\bar{3}\cdot1674+1\cdot8049$ (*d*) $\bar{2}\cdot6925+\bar{3}\cdot0957$.

2. (*a*) $3\cdot4906-1\cdot9538$ (*b*) $2\cdot6386-\bar{2}\cdot9617$

 (*c*) $\bar{1}\cdot9945+2\cdot4306-0\cdot2199$ (*d*) $\bar{2}\cdot9407-2\cdot8984+1\cdot5664$.

3. (*a*) $1\cdot0759\times4$ (*b*) $\bar{2}\cdot7394\times5$

 (*c*) $\bar{2}\cdot9367\times3$ (*d*) $\bar{1}\cdot9743\times7$.

4. (*a*) $\bar{1}\cdot3232\div3$ (*b*) $\bar{1}\cdot5617\div2$

 (*c*) $\bar{1}\cdot7138\div4$ (*d*) $\bar{3}\cdot3745\div5$.

Example. Find the characteristics of the following :

(*a*) log $747\cdot3$ (*b*) log $3\cdot074$ (*c*) log $0\cdot4902$ (*d*) log $0\cdot00508$.

Solution. (*a*) The number $747\cdot3$ is greater than 1 and has *three* digits in the integral part.

$\therefore$ The characteristic$=(3-1)=2$

(*b*) The number $3\cdot074$ is greater than 1 and has *one* digit in the integral part.

$\therefore$ The characteristic$=(1-1)=0$

(*c*) The number $0\cdot4902$ is less than 1 and has *no* zero immediately after the decimal point.

$\therefore$ The characteristic$=-(0+1)=-1=\bar{1}$.

(*d*) The number $0\cdot00508$ is less than 1 and had *two* zeros immediately after the decimal point.

$\therefore$ The characteristic$=-(2+1)=-3=\bar{3}$.

The mantissa is always kept positive.

Suppose the logarithm of a number is $2\cdot5619$.

Here the decimal part *i.e.*, mantissa is *positive.* The mantissa is $\cdot5619$.

Suppose the logarithm of a number is $-2\cdot5619$.

Here the decimal part is *negative.* It should be made positive.

Then
$$\begin{aligned}-2\cdot5619&=-2-\cdot5619\\&=-2-1+(1-\cdot5619)\\&=-3+\cdot4381\end{aligned}$$

Thus characteristic$=-3$ and mantissa$=\cdot4381$.

But $-3+\cdot4381$ is written as $\bar{3}\cdot4381$. It is read as "*bar three point four, three, eight, one.*"

The horizontal line placed over 3 is called **bar** and denotes that the characteristic is *negative.*

Let log $2963=3\cdot4717$

$$\text{Then }\log 29\cdot63=\log\frac{2963}{100}=\log 2963-\log 100$$
$$=3\cdot4717-2 \qquad =1\cdot4717.$$

$$\log\cdot2963=\log\frac{2963}{10000}=\log 2963-\log(10)^4,$$
$$=3\cdot4717-4 \qquad =\bar{1}\cdot4717.$$

$$\log \cdot 02963=\log \frac{2963}{100000} \quad =\log 2963-\log (10)^5$$
$$=3\cdot 4717-5 \qquad =\bar{2}\cdot 4717.$$

Note that numbers 2963, 29·63, ·29·63, ·02963 consist of the *same* significant digits, arranged in the same order. They only differ in the position of the decimal point.

We see that the mantissae of the logarithms of all the above numbers are the *same*, but their characteristics are changed according to the position of the decimal point.

Thus, **the mantissae of the logarithms of all numbers consisting of the same digits arranged in the same order but differing only in the position of their decimal points, are the same.**

Example 1. If $\log 3682=3\cdot 5660$, write down the value of

(*a*) $\log 3\cdot 682$ (*b*) $\log \cdot 003682$.

Solution. Given $\log 3682=3\cdot 5660$

Then $\log 3\cdot 682=0\cdot 5660$

$\log \cdot 003682=\bar{3}\cdot 5660$

Example 2. If $\log 2=\cdot 3010$, find $\log \cdot 0005$.

Solution.

$$\log (\cdot 0005)=\log \frac{5}{10000} \qquad =\log \frac{1}{2000}$$
$$=\log 1-\log 2000 \qquad =0-\log (2\times 10^3)$$
$$=-(\log 2+\log 10^3) \qquad =-\log 2-3 \log 10$$
$$=-\cdot 3010-3\times 1 \qquad =-3-\cdot 3010$$
$$=-3-1+(1-\cdot 3010) \qquad =-4+\cdot 6990$$
$$=\bar{4}\cdot 6990.$$

Example 3. Find the number of digits in 3^{62} and the position of the first significant figure in 3^{-32}, given $\log 3=\cdot 4771$.

Solution. Let $x=3^{62}$ Then $\log x=\log 3^{62}$.

$$\Rightarrow \quad \log x=62 \log 3=62\times \cdot 4771$$
$$=29\cdot 5802$$

Here the characteristic is 29. So the number of digits in 3^{62} will be 29+1 *i.e.*, 30.

Let $x=3^{-32}$

Then $\log x=\log 3^{-32}$

$$\Rightarrow \quad \log x=32 \log 3 \qquad =-32\times \cdot 4771$$
$$=-15\cdot 2672 \qquad =-15-\cdot 2672$$
$$=-16+(1-\cdot 2672) \quad =\overline{16}\cdot 7328.$$

Here the characteristic is -16. So there will be 15 zeros just after the decimal point in 3^{-32}. Thus the first significant figure will be in the 16th place of decimals.

EXERCISE—8(f)

(Section—A)

1. Find the characteristic of the following :

(*a*) log 6937 (*b*) log 56·49 (*c*) log ·0752 (*d*) log 4·093

(*e*) log ·0075 (*f*) log ·000834.

2. Find the characteristic in the logarithms of the following numbers :
(*a*) 297·6 (*b*) 0·0835 (*c*) 0·0009

3. What is number of digits in the integral part of the numbers whose logarithms are
(*a*) 4·47712 (*b*) 1·48509 (*c*) 0·84509 ?

4. Give the position of the first significant digit in the numbers whose logarithms are
(*a*) $\bar{1}\cdot90309$ (*b*) $\bar{2}\cdot30103$ (*c*) $\bar{3}\cdot36173$

5. If the mantissa of the logarithms of the number 3578 is ·5537, find the logarithms of
(*i*) 35·78 (*ii*) 0·3578 (*iii*) 0·0003578.

6. Without using logarithm tables, evaluate $3+\log_{10}(10^{-2})$.

(Section—B)

7. Simplify log 4+log 5−log 2 without using the tables.

8. Simplify without using tables :
$\frac{1}{2}\log 25-2\log 3+\log 18$.

9. Without using the log tables, find the value of
$2\log 5+\log 8-\frac{1}{2}\log 4$.

10. If log 3=0·4771 and log 7=0·8457, find the value of log $(2\frac{1}{3})$.

11. If log 64439=4·8091488, find the numbers whose logarithms are
(*i*) 6·8091488 (*ii*) $\bar{1}\cdot8091488$ (*iii*) $\bar{4}\cdot8091488$.

12. Given log 2=·30103. Find the value of (*a*) log 128, (*b*) log ·128.

13. Given log 2=·30103, log 7=·84510, find the value of log 112.

14. Given that $\log_{10} 2=0\cdot3010$ and $\log_{10} 3=0\cdot4771$. Without using tables, calculate the value of
(*a*) $\log_{10} 36$ (*b*) $\log_{10} 80$.

(Section—C)

15. Find the number of digits in 2^{100}, being given that $\log_{10} 2=\cdot30103$.

16. Given log 3=·4771, find the number of digits in (*i*) 3^{45}, and the position of the first significant figure in (*ii*) 3^{-57}.

8.8. USE OF LOGARITHM TABLES

Different tables have been prepared for the mantissae correct to different places of decimals—four, five and seven. For all practical purposes the four figure-tables are used. These tables give the mantissae of the logarithms of all numbers consisting of *four* significant digits correct to *four* places of decimals. The decimal points are dropped in the tables for convenience in the printing.

Let us now learn how to consult the table of logarithms for finding the mantissae.

Four-figure tables are given at the end of the book. A part of it is given below to explain its use.

	0	1	2	3	4	5	6	7	8	9	Mean difference								
											1	2	3	4	5	6	7	8	9
65	8129								8182								5		
66	8195								8248								5		
67	8261								8312								5		
68	8325								8376								4		
69	8388								8439								4		
70	8451								8500								4		

(*a*) We see a column on the extreme left containing numbers of *two* digits. The two digits given in this column represent the *first two significant digits* of the given number whose logarithm is required.

(*b*) Then we see *ten* columns headed by the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively. These digits represent the *third significant digit* of the given number.

(*c*) We see *nine* more columns headed by the digits, 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively under the heading '*mean differences*'. These digits represent the *fourth significant digit* of the given number.

(*i*) To find the logarithm of 69

First we see 69 in the column on the extreme left. Opposite to it, we find 8388 in the next column headed by 0. So the mantissa of log 69 is ·8388. As the given number has only *two* digits, its characteristic is 1.

∴ log 69=1·8388.

(*ii*) To find the logarithm of 698

In the extreme left-hand column, we see the number 69. Now we see further in the same horizontal row beginning with 69 till we find the number 8439 in the column headed by 8. Then ·8439 is the mantissa for the logarithm of 698. But its characteristic is 2.

∴ log 698=2·8439.

(*iii*) To find the logarithm of 6987

First we see the mantissa corresponding to 698 as above. Now we find 8439 and then we see further in the *same* horizontal row till we find 4 in the small column headed by 7 under mean differences. Adding 4 to 8439, we get 8443. Then the mantissa of log 6987 is ·8443 and its characteristic is 3.

∴ log 6987=3·8443.

Remember the following points while using tables.

(1) *The mantissae of log 2, log 3, log 4, etc., are the same as those of log 20, log 30, log 40, etc.*

(2) *The decimal point is not given before any number in logarithm tables. So always place a decimal before the obtained mantissa.*

(3) *While finding mantissa ignore the decimal point of the given number and consider only the significant figures.*

8.9 ANTILOGARITHMS

The number corresponding to a given logarithm is called its **antilogarithm**. In short notation we write antilogarithm as *antilog*.

If $\log m = x$, then m is called the antilogarithm of x *and* is written as '*antilog x*'.

e.g., $\log 47{\cdot}86 = 1{\cdot}6799$, then antilog $1{\cdot}6799 = 47{\cdot}86$

When the logarithm of a number is given, the number can be found from the table of logarithms by reversing the process explained for logarithm. But antilog tables have also been provided for finding the numbers. The tables of antilogarithms are used the same way as the tables of logarithms.

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
·71						5188											8		
·72						5309											9		
·73						5433											9		
·74						5559											9		
·75						5689											9		

(1) First column to the extreme left represents the *first two digits* of the decimal part of the given logarithm.

(2) Each of the next ten columns represents the *third digit* of the decimal part of the given logarithm.

(3) Then, each of the nine mean differences columns represents the *fourth digit* in the decimal part of the given logarithm.

To find antilog $\bar{3}{\cdot}7457$

First we see ·74 in the extreme left-hand column. Now we see in the same horizontal line till we find 5559 in the vertical column headed by 5. Then we see further in the same horizontal line till we find 9 in the small vertical column headed by 7 under mean differences.

Here characteristic is $\bar{3}$. So there will be two zeros after the decimal point.

Then antilog ·745 = 5559

Mean difference for 7 = 9

Antilog ·7457 = 5568

$\therefore$ Antilog $\bar{3}{\cdot}7457 = {\cdot}005568$.

While using antilogarithm tables we only consider mantissa. From the characteristic, we find the number of digits in the integral part, or number of zeros immediately after the decimal point in the required number.

EXERCISE—8 (g)

(Section—A)

1. Find the value of the following :
   (*a*) log 478 (*b*) log 9·35 (*c*) log ·652 (*d*) log 81·59
   (*e*) log ·07583 (*f*) log ·0003471.

2. Find the logarithms of the following numbers :
   (*a*) 6820 (*b*) 2·008 (*c*) ·00007832.

(Section—B)

3. Find the value of x in each of the following :
   (*a*) $\log x = 2{\cdot}7701$ (*b*) $\log x = 1{\cdot}7604$ (*c*) $\log x = 0{\cdot}5401$ (*d*) $\log x = \bar{2}{\cdot}7619$

4. Find the numbers whose logarithms are
   (*a*) 0·6799 (*b*) $\bar{2}{\cdot}7257$ (*c*) 5·5739 (*d*) 1·9568
   (*e*) $\bar{3}{\cdot}6808$ (*f*) 1·8455.

5. Simplify the following and find the value of x :
   (*a*) $\log x = \bar{1}{\cdot}862 - \bar{4}{\cdot}891 - 2{\cdot}761$.

6. Given $\log x = 1{\cdot}4713$, $\log y = 0{\cdot}3880$ and $\log z = \bar{1}{\cdot}7266$, use mathematical tables to find the values of
   (*a*) x (*b*) y^2 (*c*) $\sqrt{z}$.

8.10. USE OF LOGARITHMS

Example 1. Find the value of $7{\cdot}8 \times 4{\cdot}325$.

Solution. Let $x = 7{\cdot}8 \times 4{\cdot}325$

Then

$$\begin{aligned} \log x &= \log (7{\cdot}8 \times 4{\cdot}325) \\ &= \log 7{\cdot}8 + \log 4{\cdot}325 \\ &= 0{\cdot}8921 + 0{\cdot}6360 \\ &= 1{\cdot}5281 \end{aligned}$$

Now we find antilog of 1·5281.

$$\begin{aligned} \text{antilog } {\cdot}528 &= 3373 \\ \text{difference for } 1 &= \quad\;\, 1 \\ \hline \text{antilog } {\cdot}5281 &= 3374 \end{aligned}$$

$$\therefore \quad \text{antilog } 1{\cdot}5281 = 33{\cdot}74$$

$$\therefore \quad x = 33{\cdot}74.$$

Example 2. Divide 42·87 by 0·00358.

Solution. Let $x = \dfrac{42{\cdot}87}{0{\cdot}00358}$

Then

$$\begin{aligned} \log x &= \log \frac{42{\cdot}87}{0{\cdot}00358} \\ &= \log 42{\cdot}87 - \log 0{\cdot}00358 \\ &= 1{\cdot}6321 - \bar{3}{\cdot}5539 \\ &= 1{\cdot}6321 - (-3 + {\cdot}5539) \\ &= 1{\cdot}6321 + 3 - {\cdot}5539 \end{aligned}$$

$$=4{\cdot}0782$$

$$\log x=\log 11980$$

Hence $x=11980$

Here the characteristic is 4. So the number consists of 5 digits and hence add one zero in the end.

EXERCISE—8(h)

(Section—A)

Using the tables of the logarithms and antilogarithms, find the values of the following :

1. $43{\cdot}5\times75{\cdot}61$.
2. $27{\cdot}49\times0{\cdot}0469$.
3. $0{\cdot}0287\times0{\cdot}000372$.
4. $9{\cdot}753\times10{\cdot}34\times0{\cdot}9252$.
5. $43{\cdot}54\div12{\cdot}67$.
6. $13{\cdot}67\div0{\cdot}0472$.
7. $\sqrt[3]{0{\cdot}1437}$.
8. $\sqrt[3]{0{\cdot}05358}$.
9. $\sqrt[5]{{\cdot}00000165}$.

(Section—B)

Use logarithms to find the value of

10. $\dfrac{(21{\cdot}2)^2}{300}$.
11. $\dfrac{(10{\cdot}23)^3}{502{\cdot}3}$.
12. $\sqrt{27{\cdot}23\times3{\cdot}276}$.
13. $(36{\cdot}71\times0{\cdot}8246)\div546$.
14. $2{\cdot}821\times0{\cdot}699\div24{\cdot}3$.

Example. Evaluate $\dfrac{(6{\cdot}284)^3.\ \sqrt[3]{624}}{({\cdot}005)^{1/4}}$

Solution. Let $x=\dfrac{(6{\cdot}284)^3.\ \sqrt[3]{624}}{({\cdot}005)^{1/4}}$

Then $\log x=\log\left[\dfrac{(6{\cdot}284)^3\times(624)^{1/3}}{({\cdot}005)^{1/4}}\right]$

$$=\log(6{\cdot}284)^3+\log(624)^{1/3}-\log({\cdot}005)^{1/4}$$

$$=3\log 6{\cdot}284+\tfrac{1}{3}\log 624-\tfrac{1}{4}\log{\cdot}005$$

$$=3\,(0{\cdot}7983)+\tfrac{1}{3}\,(2{\cdot}7952)-\tfrac{1}{4}\,(\bar{3}{\cdot}6990)$$

$$=2{\cdot}3949+0{\cdot}9317-\tfrac{1}{4}\,(-3+{\cdot}6990)$$

$$=3{\cdot}3266+1-{\cdot}4247$$

$\Rightarrow$ $\log x=3{\cdot}9019$

Antilog $3{\cdot}9019=7979$

$\therefore$ $x=7979$.

EXERCISE—8(i)

(Section—A)

Use logarithms to find the value of

1. $\dfrac{18{\cdot}29\times\sqrt{741}}{4{\cdot}05}$ ·
2. $15{\cdot}28\times\sqrt[3]{\dfrac{501{\cdot}4}{0{\cdot}023}}$
3. Use logarithm tables to evaluate $\sqrt{\dfrac{0{\cdot}732}{(15{\cdot}6)^2}}$
4. Find the value of $\dfrac{72{\cdot}2\times281{\cdot}8}{42{\cdot}1\times{\cdot}0065}$.

(Section—B)

5. Simplify with the use of four figure tables

$$(6\cdot4)^{1/10}\times\sqrt[4]{(256)^3}\div\sqrt{80}.$$

6. Find the value of $\sqrt{\dfrac{93\cdot7}{5\cdot07\times0\cdot0624}}$

7. Use logarithms to find the value of $\dfrac{16\cdot28\times\sqrt{0\cdot652}}{5\cdot05}$ correct to 3 significant figures.

8. Evaluate $\dfrac{(4\cdot321)^2+(8\cdot468)^2}{\sqrt{0\cdot3645}}$

9. Calculate $\sqrt[3]{\dfrac{0\cdot7214\times20\cdot37}{69\cdot8}}$

10. Use logarithms to find the value of $\dfrac{(0\cdot312)^2\times12\cdot4}{0\cdot1234}$ correct to 2 significant figures.

(Section—C)

11. Evaluate $\dfrac{\sqrt{0\cdot9516}\ \sqrt[6]{0\cdot04364}}{\sqrt[4]{0\cdot003123}}$

12. Evaluate $\dfrac{(\cdot0437)^{2/5}\times(1\cdot407)^2}{(\cdot0015)^{1/2}\times(1\cdot235)^{1/7}}$

13. Using four-figure tables, find the value of $\dfrac{(6\cdot45)^3\times\sqrt[3]{\cdot00034}}{(9\cdot37)^2\times\sqrt[4]{8\cdot93}}$

14. Compute $\dfrac{(21\cdot65)^{3/4}\times(0\cdot75)^{2/7}}{(12\cdot56)^{2/9}\times(75\cdot18)^{4/5}}$

15. Find the value of $\left[\dfrac{8^{1/3}\times7^{3/4}}{(7\cdot29)^{1/3}\times3\cdot26}\right]^{1/2}$ correct to four places of decimals.

8.11. APPLICATIONS

We have studied about compound interest, population growth, depreciation of value of an article, areas of rectangles, squares, triangles, rhombuses, trapeziums, parallelograms, etc., in earlier classes. We shall now solve harder problems based on these, using logarithmic tables.

If P is the principal, n the number of periods, r the rate of interest per cent per period, and A the amount at the end of periods, then A is given by the formula

$$A=P\left(1+\frac{r}{100}\right)^n$$

Given any three of the variables P, r, n and A, the fourth is easily determined using logarithmic tables.

Example 1. Find the compound interest on Rs. 1600 at 8% per annum for 2 years when the interest is compounded half-yearly.

Solution. Here $P=$ Rs. 1600, $r=8\%$ per annum $=4\%$ per half-year
$n=2$ years $=4$ half years.

Using the formula $A=P\left(1+\frac{r}{100}\right)^n$, we get

$$A=1600\left(1+\frac{4}{100}\right)^4$$

or $$A=1600\ (1+0\cdot04)^4$$

or $$A=1600\times(1\cdot04)^4$$

$$\therefore \quad \log A=\log\ [1600\times(1\cdot04)^4]$$
$$=\log 1600+\log\ (1\cdot04)^4$$
$$=\log 1600+4 \log 1\cdot04^4$$
$$=3\cdot2041+4\times0\cdot0170$$
$$=3\cdot2041+0\cdot0680$$
$$=3\cdot2721.$$

$$\therefore \quad A=\text{antilog } 3\cdot2721=1871$$

Thus compound interest = Amount − Principal
= Rs. 1871 − Rs. 1600
= Rs. 271.

If we calculate compound interest directly without using log tables, the result would be Rs. 271·77. This difference of 77 paise is due to the fact that the mantissae obtained from log tables are *approximate values.*

When an entity increases in magnitude over a period of time, we say that it has *grown* during that period. The *growth* is measured as a ratio of the final magnitude to the initial magnitude.

Very often we talk of *rate of growth* of money, population, per capita income, weights and individuals, etc. The rate of growth of objects can be *positive* in some cases, as in case of population ; in some other cases it can be *negative* as in the case of depreciation of machinery over a period of time.

We are familiar with the compound interest formula used in the above example. The same formula is applicable in problems on rate of growth.

Population A after n years with growth rate at $r\%$ per annum is given by the formula.

$A=P\left(1+\frac{r}{100}\right)^n$, where the present population is P.

Value A after n years of a machinery with depreciation at the rate of $r\%$ per annum is given by the formula

$A=P\left(1-\frac{r}{100}\right)^n$ where P is the present value of the machinery.

Example 2. The industrial output of a scooter factory increases by 5% of what it had been in the beginning of every year. Find the production after 3 years, if the factory turns out 80,000 scooters per year.

Solution. Here $P=80{,}000$, $r=5\%$, $n=3$.

Using the formula $A=P\left(1+\frac{r}{100}\right)^n$, we get

Production after 3 years, $A=80{,}000\left(1+\frac{5}{100}\right)^3$
$$=80{,}000\ (1+\cdot05)^3$$
$$=80{,}000\times(1\cdot05)^3$$

$$\begin{aligned}
\therefore \quad \log A &= \log 80{,}000 + \log (1{\cdot}05)^3 \\
&= 4{\cdot}9031 + 3 \log 1{\cdot}05 \\
&= 4{\cdot}9031 + 3 \times (0{\cdot}0212) \\
&= 4.9031 + 0{\cdot}0636 \\
&= 4{\cdot}9667 \\
\therefore \quad A &= \text{antilog } 4{\cdot}9667 \\
&= 92{,}620.
\end{aligned}$$

Hence production after 3 years=92,620 scooters.

Example 3. A machinery plant costing Rs. 1,60,000 depreciates each year by 10% of its value at the beginning of the year. Find the value of the plant after 5 years.

Solution. Cost of the plant $(P)=$Rs. 1,60,000
Rate of depreciation $(r)=10\%$
Period of time $(n)=5$ years.

Let the depreciated value of the plant after 5 years be Rs. x.

Using the formula $A=P\left(1-\frac{r}{100}\right)^n$, we get

$$\begin{aligned}
x &= 1{,}60{,}000\left(1-\frac{10}{100}\right)^5 \\
&= 1{,}60{,}000\,(1-0{\cdot}1)^5 \\
&= 1{,}60{,}000 \times (0{\cdot}9)^5 \\
\therefore \quad \log x &= \log 1{,}60{,}000 + \log (0{\cdot}9)^5 \\
&= \log 1{,}60{,}000 + 5 \log 0{\cdot}9 \\
&= 5{\cdot}2041 + 5 \times (\bar{1}{\cdot}9542) \\
&= 5{\cdot}2041 + 5\,(-1 + 0{\cdot}9542) \\
&= 5{\cdot}2041 - 5 + 4{\cdot}7710 \\
&= 4{\cdot}9751 \\
\therefore \quad x &= \text{antilog } 4{\cdot}9751 \\
&= 94430.
\end{aligned}$$

Hence the depreciated value of the plant=Rs. 94,430.

Let us recall the following formulae for the areas of plane figures :

(1) Area of a rectangle=length × breadth sq. units.

(2) Area of a square $=(\text{side})^2$ sq. units.

(3) Area of a triangle $=\frac{1}{2}$ (base × altitude) sq. units.

(4) Area of an equilateral triangle$=\frac{\sqrt{3}}{4}(\text{side})^2$ sq. units.

(5) Area of an isosceles triangle $=\frac{a}{4}\sqrt{4b^2-a^2}$, where sides are a, b and b.

(6) Area of a triangle$=\sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are lengths of sides and $s=\frac{a+b+c}{2}$

(7) Area of a parallelogram = base × height sq. units.

(8) Area of a rhombus $=\frac{1}{2}$ (product of the diagonals), sq. units

(9) Area of a trapezium $=\frac{1}{2}$ (sum of parallel sides) × height, sq. units

(10) Area of a quadrilateral $=\frac{1}{2}$ (sum of its off sets) × diagonal, sq. units

Example 4. Find the area of a triangle whose sides are 14 2 dm, 12·6 dm and 8·4 dm.

Solution. Let $a=14{\cdot}2$ dm, $b=12{\cdot}6$ dm, $c=8{\cdot}4$ dm.

Then $$s=\frac{a+b+c}{2}=\frac{14{\cdot}2+12{\cdot}6+8{\cdot}4}{2}\text{ dm}$$
$$=17{\cdot}6\text{ dm}$$
$$s-a=(17{\cdot}6-14{\cdot}2)\text{ dm.}=3{\cdot}4\text{ dm}$$
$$s-b=(17{\cdot}6-12{\cdot}6)\text{ dm.}=5\text{ dm}$$
$$s-c=(17{\cdot}6-8{\cdot}4)\text{ dm.}=9{\cdot}2\text{ dm}$$

Using the formula $A=\sqrt{s(s-a)(s-b)(s-c)}$, we get
$$A=\sqrt{17{\cdot}6\times3{\cdot}4\times5\times9{\cdot}2}\text{ sq. dm}$$

$\therefore$ $$\log A=\log(17{\cdot}6\times3{\cdot}4\times5\times9{\cdot}2)^{1/2}$$

or $$\log A=\frac{1}{2}\log(17{\cdot}6\times3{\cdot}4\times5\times9{\cdot}2)$$
$$=\frac{1}{2}[\log 17{\cdot}6+\log 3{\cdot}4+\log 5+\log 9{\cdot}2]$$
$$=\frac{1}{2}[1{\cdot}2455+0{\cdot}5315+0{\cdot}6990+0{\cdot}9638]$$
$$=\frac{1}{2}[3{\cdot}4398]$$
$$=1{\cdot}7199$$

$\therefore$ $$A=\text{antilog }1{\cdot}7199$$

or $$A=52{\cdot}47.$$

Hence the area of the triangle is $52{\cdot}47$ dm^2.

Example 5. What is the radius of a sphere whose volume is 288 dm^3. Give the answer correct to three decimal places, taking $\pi=\frac{22}{7}$.

Solution. Volume of a sphere $=\frac{4}{3}\pi r^3$.

But the volume of the sphere $=288$ cm^3.

Then $$\frac{4}{3}\pi r^3=288$$

or $$\frac{4}{3}\times\frac{22}{7}\times r^3=288$$

or $$\frac{88r^3}{21}=288$$

$$\therefore \quad \log 88r^3-\log 21=\log 288$$

or $$\log 88+\log r^3-\log 21=\log 288$$

or $$1{\cdot}9445+3\log r-1{\cdot}3222=2{\cdot}4594$$

or $$3\log r=2{\cdot}4594+1{\cdot}3222-1{\cdot}9445$$
$$=1{\cdot}8371$$

$$\therefore \quad \log r=0{\cdot}6124$$
$$r=\text{antilog } 0{\cdot}6124$$
$$=4{\cdot}097.$$

Hence, the radius of the sphere = 4·097 dm.

EXERCISE—8 (j)

(Section—A)

1. Using logarithm tables, find the volume of a cube whose one side is 2·255 metres.
2. Find the approximate area of a circular plot of land whose radius is 75 *dm*. (Take $\log \pi=0{\cdot}4972$).
3. Use logarithms to find the radius of a circle whose area is 48·5 sq. *m*., taking $\pi=\frac{22}{7}$.
4. In how many years will Rs 4000 amount to Rs 5324 at compound interest at 10% per annum ?
5. Taking π to be 3·142, use logarithms table to calculate to two significant figures, the radius of a sphere whose volume is 2·462 dm^3.

(Section—B)

6. Find to the nearest rupee the amount of Rs 1000 in 20 years at 4% per annum compound interest when interest is due yearly.
7. Find the compound interest on Rs. 6250 at 14% per annum for 1½ years when it is calculated half yearly.
8. A manufacturing concern write off depreciation at the rate of 5% every year of its plant and machinery, the original cost of which is Rs 3,00,000. Find the depreciated value at the end of 10 years.
9. What sum of money will amount to Rs. 8316 in 3 years at compound interest, the interest for first, second and third years being 8%, 10% and 12% respectively.
10. The sides of a triangle are 48·69 *dm*., 52·09 *dm*. and 32·8 *dm*. Find its area.
11. The volume of a cylinder whose height is equal to its radius is 2090 dm^3. Find the height of the cylinder, correct to two decimal places.
12. If the volume of a sphere is 128·1 dm^3, find its total surface area.

(Section—C)

13. In the 1981 census, the population of India was found to be $6{\cdot}7\times10^7$. If the population increases at the rate of 2·5% every year, what would be the population in 1991 ?

14. In a factory the production of scooters rose to 46,340 from 40,000 in 3 years. Find the annual rate of growth of the production of scooters.

15. The time period T (in seconds) of a simple pendulum is given by the formula

$$T=2\pi\sqrt{\frac{l}{g}},$$

where g is the acceleration due to gravity. If l is 281·3 *cm* and $g=981$ cm/sec^2, find the time period of the pendulum, taking $\pi=3{\cdot}1415$.

16. At what rate per cent compound interest will the sum of Rs. 3350 amount to Rs 5535 in 3 years ?

17. The lengths of parallel sides of a trapezium are 56 cm. 40 cm and the length of the outer sides are 28 cm and 30 cm. Find the area of the trapezium.

18. A cubical block of metal, each edge of which is 36 4 cm, is melted down into a sphere. Find the diameter of the sphere as correctly as possible with Four-Figure Log Tables.

19. The value of a machinery plant depreciates by 11% per year. Its present value is Rs 38,400. What was its value 3 years ago ?

20. From a point in the interior of an equilateral triangle perpendiculars drawn to the three sides are 84 *cm*, 49 *cm* and 44 *cm* respectively. Find the area of the triangle.

REVIEW EXERCISE—VIII

(Section—A)

1. In answer to each of the following questions write 'True' or 'False' as appropriate :

   (*a*) If $\log_{10} x=a$, then $10^a=x$

   (*b*) $\log 1\times\log 1000=0$

   (*c*) $\dfrac{\log x}{\log y}=\log x-\log y$

   (*d*) The mantissa of the logarithm of 833·4 to the base 10 is 2.

2. If $\log x=-3$, find the value of x.

3. Given $\log 4{\cdot}113=0{\cdot}6142$. Find x, if $\log x=2{\cdot}6142$.

4. Without using log tables, find the value of

$$\log_{10} 200+\log_{10} 5.$$

5. Express $\log_{10} 2+1$ in the form $\log_{10} x$.

6. If $\log 12{\cdot}45$ is $1{\cdot}0952$ and $\log_{10} 3{\cdot}79$ is $0{\cdot}5786$, find the value of

$$\log_{10} 124{\cdot}5+\log_{10} 379.$$

7. If $\log (4y-3)=\log (2y+1)-\log 3$, find the value of y without using logarithmic tables.

(Section—B)

8. Express $\log \dfrac{a^5\sqrt{b}}{\sqrt[3]{c^2}}$ in terms of $\log a$, $\log b$, $\log c$.

9. Show that $3\log_2 5+\log_2 10-\log_2 625=1$.

10. Simplyfy : $2\log \dfrac{15}{8}-\log \dfrac{25}{162}+3\log \dfrac{4}{9}$.

11. If $\log 2={\cdot}3010300$, find the number of digits in 2^{25}.

12. Find the first significant digit in $({\cdot}15)^{10}$, given $\log 2={\cdot}3010$ and $\log 3={\cdot}4771$.

13. Find the smallest index x of 7, which makes $7^x>(10)^{50}$ true, given $\log 7={\cdot}8451$.

14. Use logarithms to calculate the cube root of 0·6755.

15. Find the mean proportional between 2·87 and 30·08.

16. Using logarithms find the seventh root of 142·71.

(Section—C)

17. Find the value of $\dfrac{69{\cdot}3\times 83{\cdot}74}{\sqrt[3]{876}}$ by using logarithms.

18. Find the value of $\dfrac{\left(300\times\dfrac{1}{49}\right)}{\sqrt[3]{22\times 6{\cdot}9}}$ to the nearest integer.

19. Evaluate $\dfrac{1+\sqrt[3]{0\cdot0075}}{1-\sqrt[3]{0\cdot0075}}$.

20. Find the compound interest on Rs 1159 for 5 years at 6% compounded semi-annually.

21. A machine purchased for Rs 10,000 depreciates at the rate of 6% per annum. The depreciation is worked on the value of the machine at the beginning of the year. Use log tables to find its depreciated value, correct to nearest rupee, after 7 years.

22. The volume of a sphere is 33·87 cubic cm. Find the radius of the sphere using logarithms.

23. The time period T (in seconds) of a simple pendulum is given by the formula

$$T=2\pi\sqrt{\frac{l}{g}}.$$

Find the value of g when $l=100$ cm, $T=2$ seconds and $\pi=3\cdot142$.

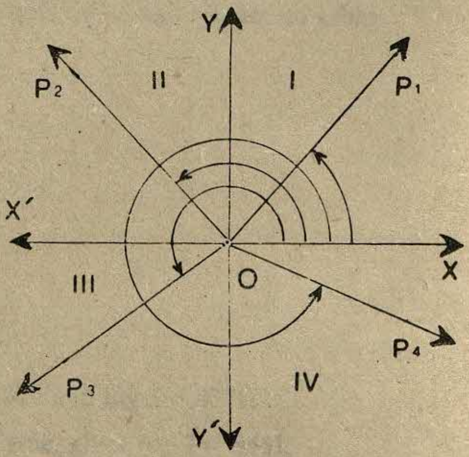
24. The volume of a sphere is 33·87 dm^3. Find the radius of the sphere, correct to three decimal places, using logarithms.

9

TRIGONOMETRY

9.1. The word "**trigonometry**" is derived from two Greek words that mean "*triangle measurement*". Originally it dealt with the angles and sides of a triangle and various relations existing between them. But now it has a much wider scope and deals with angles of any magnitude. It has also a wide application in every branch of Higher Mathematics whether pure or applied. It is of practical use to the engineers, the surveyors and the astronomers.

9.2. Euclid defined an angle as the inclination between two intersecting lines. According to this definition, an angle cannot be greater than 2 right angles. But in Trigonometry an angle can be of any magnitude and is also capable of sign—positive or negative.



Let $X'OX$ and YOY' be any two straight lines in a plane, intersecting at right angles at O.

Let a revolving line OP, starting from its initial position OX, revolve about O in the anticlockwise direction.

When the revolving line takes up the position OP_1, the angle described is XOP_1, which is *less* than a right angle.

When the revolving line takes up the position OP_2, the angle described is XOP_2, which is *greater* than a right angle.

When the revolving line takes up the position OP_3, the angle described is XOP_3, which is *greater* than two right angles.

When the revolving line takes up the position OP_4, the angle described is XOP_4, which is *greater* than three right angles.

When the revolving line comes to its initial position, *i.e.*, makes one complete revolution, the angle described is four right angles.

If the revolving line OP revolves still further and takes up the position OP_1, for the second time, after having made a complete revolution, the angle described is 4 right angles plus $\angle XOP_1$ and *not* just $\angle XOP_1$.

Similarly, if the revolving line OP starting from OX makes two complete revolutions then takes up the position OP_1, the angle described is 8 right-angles plus $\angle XOP_1$; and on.

hus a revolving line may make any number of complete revolutions before taking up its final position.

So an angle can be of any magnitude and it is measured by the amount of rotation of a revolving line about a fixed point from a fixed initial position to another position.

If the revolving line starts from its initial position and revolves in the counter clockwise direction, is described a positive angle.

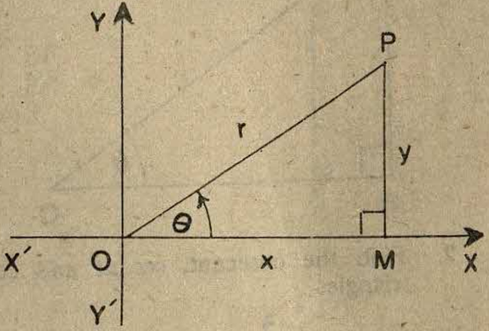
9.3. DEFINITIONS

Let XOX' and YOY' be the axes of co-ordinates.

Let a revolving line OP of length r, starting from its initial position OX, revolve through an angle θ and take up the position OP.

Draw PM perpendicular to XOX' intersecting it at M. Now we get a right-angled triangle POM.

Then side OP *(opposite to right angle) is hypotenuse*, side MP *(opposite to angle* θ) *is* **perpendicular** and side OM *(adjacent to angle* θ) is **base.**



Let the coordinates of P be (x, y).

Thus $OM=x$, $MP=y$ and $OP=r$

$\frac{y}{r}$ or $\frac{perp}{hyp.}$ is called the **sine of angle** θ.

It is written as $\sin\theta=\frac{y}{r}$

$\frac{x}{r}$ or $\frac{\text{base}}{hyp.}$ is called the **cosine of angle** θ.

It is written as $\cos\theta=\frac{x}{r}$

$\frac{y}{x}$ or $\frac{perp.}{\text{base}}$ is called the **tangent of angle** θ.

It is written as $\tan\theta=\frac{y}{x}$

$\frac{r}{y}$ or $\frac{hyp.}{perp.}$ is called the **cosecant of angles** θ.

It is written as $\text{cosec}\,\theta=\frac{r}{y}$

$\frac{r}{x}$ or $\frac{hyp.}{\text{base}}$ is called the **secant of angle** θ.

It is written as $\sec\theta=\frac{r}{x}$

$\frac{x}{y}$ or $\frac{\text{base}}{perp.}$ is called the **cotangent of angle** θ.

It is written as $\cot\theta=\frac{x}{y}$

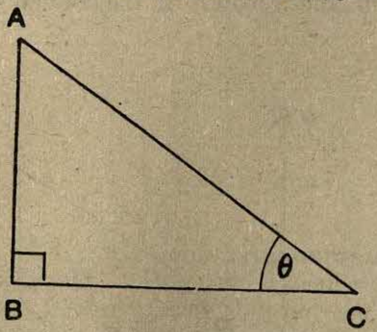
These six trigonometrical ratios are also called **Trigonometrical function** or **Circular functions** of θ.

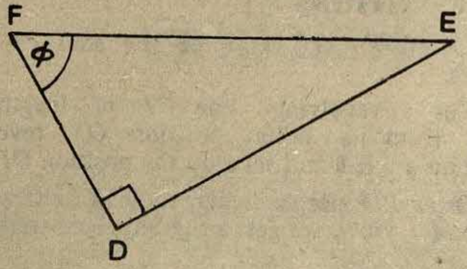
Sin θ is a single symbol and sin cannot be detached from θ. It is not equal to sin × θ *i.e.*, sin is not a multiplier of θ. A similar remark is true for other ratios also.

EXERCISE 9 (a)

(Section—A)

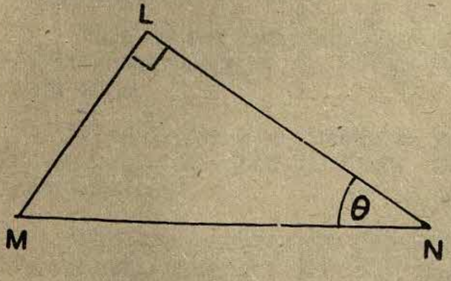
1. Find the sine, cosine and tangent of each angle marked in the following triangles :



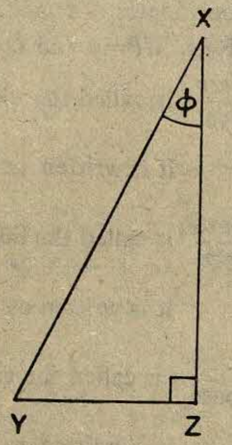


2. Find the cosecant, secant and cotangent of each angle marked in the following triangles :

(*a*)



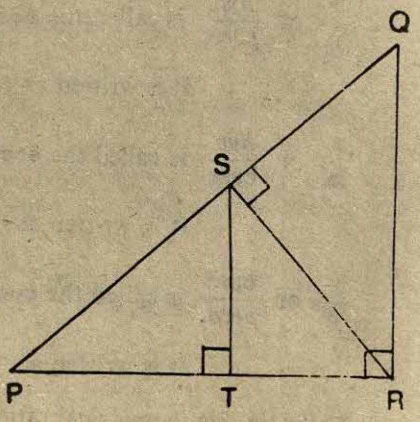
(*b*)

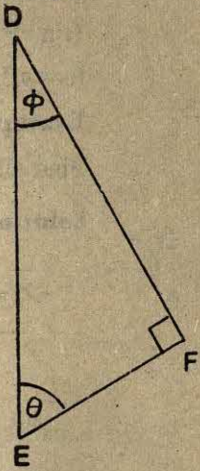


(Section—B)

3. In the accompanying figure you have three right-angled triangles *PQR*, *PST* and *QRS*.

Write the expressions for tan *QPR* sec *PST* and sin *SQR*.





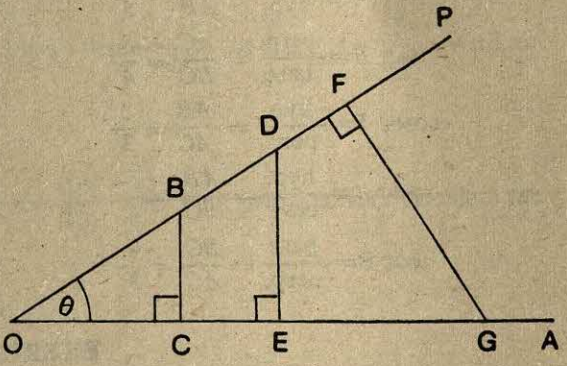
4. For the given $\triangle DEF$, write the following trigonometrical ratios in two ways.

(*a*) $\frac{EF}{DE}$ (*b*) $\frac{DE}{DF}$ (*c*) $\frac{DF}{EF}$

9.4. Let *AOP* by any acute angle. Take any point *B* and *D* on *OP*.

Draw *BC* and *DE* perpendicular to *OA*.

Also take any point *F* on *OP* and draw *FG* perpendicular at *F* intersecting *OA* at *G*.



From the right-angled triangle *BOC*,

$$\sin BOC = \frac{CB}{OB}$$

From the right-angled triangle *DOE*, $\sin DOE = \frac{ED}{OD}$

From the right-angled triangle *FOG*, $\sin FOG = \frac{FG}{OG}$

But $\angle BOC = \angle DOE = \angle FOG = \theta$ (say)

Then $\sin \theta = \frac{CB}{OB}$, $\sin \theta = \frac{ED}{OD}$, $\sin \theta = \frac{FG}{OG}$.

But the triangles *BOC*, *DOE* and *GOF*, are equiangular.

So they are similar, $\therefore \frac{CB}{OB} = \frac{ED}{OD} = \frac{FG}{OG}$.

Thus the sine of the angle *POA* is the same whether it is obtained from the $\triangle BOC$, or from the $\triangle DOE$, or from the $\triangle GOF$.

Thus trigonometrical ratios do not depend upon the length of revolving line, but depend only upon the magnitude of the angle. The trigonometrical ratios are always the *same* for the same angle.

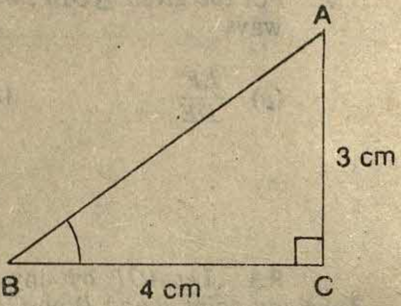
The trigonometrical ratios of an angle are numerical quantities. Each one of them is a ratio of two lengths. Therefore, like other numerical quantities, algebraical operations, addition, subtraction, multiplication, division and indices, etc., can be applied to these ratios.

$(\sin\theta)(\sin\theta)$ or $(\sin\theta)^2$ is written as $\sin^2\theta$. It is read as '*sine square* θ'.

$(\sin\theta)^3$ is written as $\sin^3\theta$. It is read as '*sine cube* θ'.

$(\cos\theta)^3$ is written as $\cos^3\theta$. It is read as '*cosine cube* θ'.

Example. In triangle ABC, $BC=4$ cm, $AC=3$ cm and $\angle C$ is a right angle. Find all trigonometrical ratios of angle B.



Solution. In $\triangle ABC$, $\angle C=90°$.

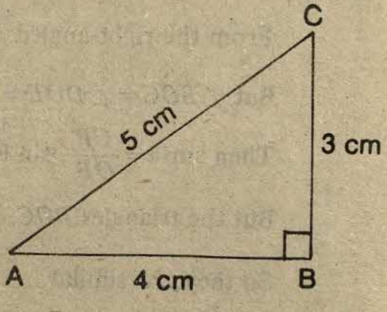
$$AB^2=AC^2+BC^2$$
$$\Rightarrow AB^2=9+16$$
$$\Rightarrow AB^2=25$$
$$\therefore AB=5 \text{ cm}$$

$$\sin B=\frac{\text{perp.}}{\text{hyp.}}=\frac{AC}{AB}=\frac{3}{5}$$
$$\cos B=\frac{\text{base}}{\text{hyp.}}=\frac{BC}{AB}=\frac{4}{5}$$
$$\tan B=\frac{\text{perp.}}{\text{base}}=\frac{AC}{BC}=\frac{3}{4}$$
$$\text{cosec } B=\frac{\text{hyp.}}{\text{perp.}}=\frac{AB}{AC}=\frac{5}{3}$$
$$\sec B=\frac{\text{hyp.}}{\text{base}}=\frac{AB}{BC}=\frac{5}{4}$$
$$\cot B=\frac{\text{base}}{\text{perp.}}=\frac{BC}{AC}=\frac{4}{3}$$

EXERCISE 9(b)

(Section—A)

1. From the adjoining figure, find the values of $\sin A$, $\cos C$, $\tan A$, $\cot C$, $\text{cosec } C$ and $\sec A$.

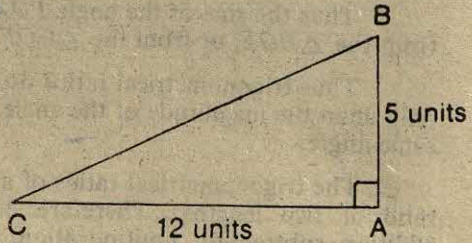


2. From the adjoining figure, find the values of

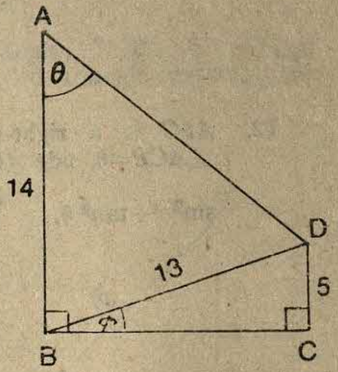
(*i*) $\sin C$ (*ii*) $\cos B$

(*iii*) $\tan B$ (*iv*) $\sec C$

(*v*) $\text{cosec } B$ (*vi*) $\cot C$



(Section—B)



3. Using the measurements given in the figure, find the value of

(*a*) sin ϕ (*b*) tan θ

4. In a right-angle triangle, it is given that A is an acute angle and that $\tan A = \frac{3}{4}$. Without using tables, find the value of cos A.

5. In a right-angled triangle it is given that A is an acute angle and $\tan A = \frac{5}{12}$. Without using tables, find the value of

(*i*) cos A (*ii*) sin A

6. Triangle ABC is right-angled at C and $\cos A = \frac{12}{13}$. Without using tables, calculate the value of sin A and tan A.

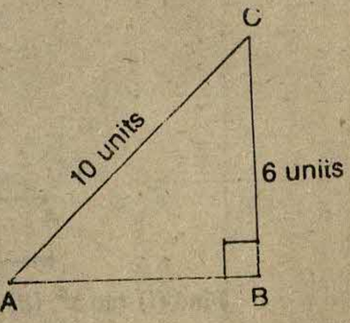
7. If $\sin \theta = \frac{3}{5}$ and θ is an acute angle, find

(*i*) cos θ (*ii*) tan θ

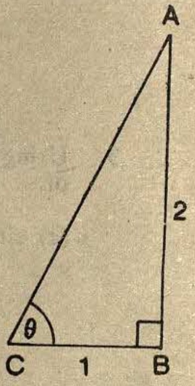
If $\tan \theta = \frac{5}{12}$, $0° < \theta < 90°$, find the values of sin θ and sec θ. [*C.B.S.E., 1987* (*Delhi*)

9. In a right-angled triangle, it is given that angle A is an acute angle such that $\tan A = \frac{5}{12}$. Find the value of cosec A − cot A.

10. Given $\tan A = \frac{15}{2}$, $0° \leqslant A < 90°$, find the value of sin A cos A. [*C.B.S.E., 1986* (*A.I.*)]



11. In the adjoining figure, $\angle B = 90°$. Find the value of $\sin^2 A + \cos^2 A$.



12. ABC is a right-angled triangle, right-angled at B. Given that $\angle ACB=\theta$, side $AB=2$ units and side $BC=1$ unit, find the value of

$\sin^2\theta+\tan^2\theta$.

13. If $\sin A=\frac{7}{25}$, find the values of $\tan A$ and $\sec A$.

14. If $\sec A=\frac{41}{9}$, find the values of $\tan A$ and $\sin A$.

15. If $\sin A=\frac{m}{n}$, prove that $\tan A=\frac{m}{\sqrt{n^2-m^2}}$.

16. If $\cos\theta=\frac{2t}{1+t^2}$, find the values of $\operatorname{cosec}\theta$ and $\cot\theta$.

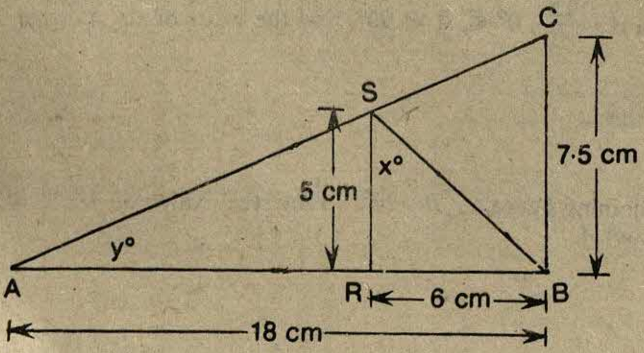
(Section—C)

17. If $5\tan\theta=4$, find the value of $\frac{5\sin\theta-3\cos\theta}{5\sin\theta+2\cos\theta}$ [*C.B.S.E., 1981 (Delhi)*]

18. If $\sin A=\frac{4}{5}$, prove that $\tan A+\sec A=3$, when A is an acute angle.

19. In a triangle ABC, right-angled at C, prove that $\tan A+\tan B=\frac{c^2}{ab}$.

20. In the figure given below, $\triangle ABC$ is right-angled at B, $\triangle BSC$ is right-angled at S, and $\triangle BRS$ is right-angled at R, $AB=18$ cm, $BC=7{\cdot}5$ cm, $RS=5$ cm, $\angle BSR=x^\circ$ and $\angle SAB=y^\circ$.

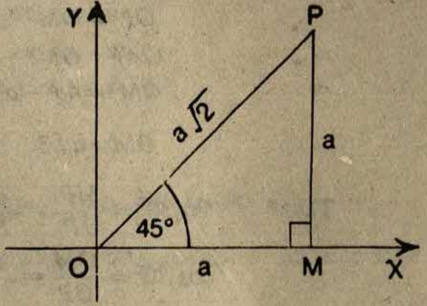


Find (*i*) $\tan x^\circ$, (*ii*) $\sin y^\circ$, (*iii*) $\cos y^\circ$.
(Leave your answers as fracticos).

9.5 TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES

Here we shall find the values of trigonometrical ratios of angles 0°, 30°, 45°, 60° and 90°. These values can be obtained with the help of a right-angled triangle containing the particular angle.

To find the trigonometrical ratios of 45°.



Let the revolving line OP, starting from OX, trace out $\angle XOP = 45°$, in the positive direction.

From P, draw $PM \perp OX$.

Now in $\triangle MOP$,

$\angle MOP = 45°$, $\angle PMO = 90°$.

Since the three angles of a triangle are together equal to 2 *rt.* $\angle s$,

$\angle OPM = 180° - 45° - 90° = 45°$

$\therefore \angle MOP = \angle OPM \quad \therefore MP = OM = a$ (say)

In the *rt.* $\angle d$ $\triangle MOP$,

$$OP^2 = OM^2 + MP^2 \qquad \text{(Pythagoras Theorem)}$$

$$\Rightarrow \quad OP^2 = a^2 + a^2 \qquad = 2a^2$$

$$\Rightarrow \quad OP = a\sqrt{2}.$$

Then

$$\sin 45° = \frac{MP}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{OM}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

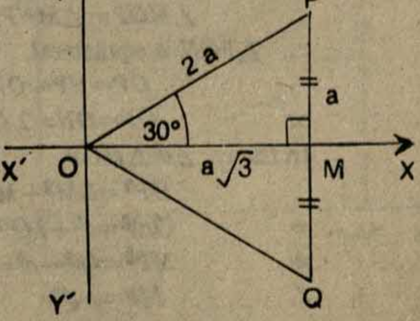
$$\tan 45° = \frac{MP}{OM} = \frac{a}{a} = 1$$

$$\text{cosec } 45° = \frac{OP}{MP} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\sec 45° = \frac{OP}{OM} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\cot 45° = \frac{OM}{MP} = \frac{a}{a} = 1.$$

To find the trigonometrical ratios of 30°.



Let the revolving line OP, starting from OX, trace out $\angle XOP = 30°$, in the positive direction.

From P, draw $PM \perp OX$.

Produce PM to Q such that $MQ = PM$.

Join OQ.

In $\triangle OMP$ and $\triangle OMQ$,

$MP = MQ$

$OM = OM$ (common)

Incl. $\angle OMP$ = Incl. $\angle OMQ$ = 1 rt $\angle$.

$\therefore \quad \triangle OMP \cong \triangle OMQ$

$\therefore \quad \angle MPO = \angle MQO$

But $\angle MPO = 180^\circ - (30^\circ + 90^\circ) = 60^\circ$

$\therefore$ $\angle MPO = \angle MQO = 60^\circ$

$\therefore$ $\triangle POQ$ is an equilateral triangle.

$\therefore$ $OP = OQ = PQ$

or $OP = PQ = 2MP = 2a$ (say)

In the rt. $\angle d$ $\triangle MOP$

$OP^2 = OM^2 + MP^2$ (Pythagoras Theorem)

$\Rightarrow$ $(2a)^2 = OM^2 + a^2$

$\Rightarrow$ $OM^2 = 4a^2 - a^2 = 3a^2$

$\therefore$ $OM = a\sqrt{3}$ $\left\{\begin{array}{l}\text{Taking} + \text{ve sign with the square root as } OM \\ \text{is measuredalong } OX.\end{array}\right.$

Then

$$\sin 30^\circ = \frac{MP}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{OM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{MP}{OM} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

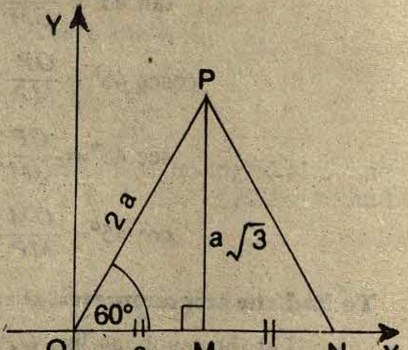
$$\operatorname{cosec} 30^\circ = \frac{OP}{MP} = \frac{2a}{a} = 2$$

$$\sec 30^\circ = \frac{OP}{OM} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\cot 30^\circ = \frac{OM}{MP} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

To find the trigonometrical ratios of 60°.

Let the revolving line OP, starting from OX, trace out $\angle XOP = 60^\circ$, in the positive direction.



From P, draw $PM \perp OX$.

From MX, cut off MN such that $MN = OM$.

Join PN.

Now in $\triangle POM$ and $\triangle PNM$

$OM = MN$

$MP = MP$ (common)

included $\angle OMP =$ included $\angle NMP = 1$ *rt.* $\angle$

$\therefore$ $\triangle POM \cong \triangle PNM$

$\therefore$ $\angle MOP = \angle MNP = 60^\circ$

$\therefore$ $\triangle PON$ is equilateral.

$\therefore$ $OP = NP = ON$

or $OP = ON = 2\,OM = 2a$ (say)

In the *rt.* $\angle d$ $\triangle MOP$,

$OP^2 = OM^2 + MP^2$ (Pythagoras Theorem)

$\Rightarrow$ $(2a)^2 = a^2 + MP^2$

$\Rightarrow$ $MP^2 = 4a^2 - a^2 = 3a^2$

$\therefore$ $MP = a\sqrt{3}$

Then, $\sin 60° = \frac{MP}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$

$$\cos 60° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60° = \frac{MP}{OM} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

$$\operatorname{cosec} 60° = \frac{OP}{MP} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\sec 60° = \frac{OP}{OM} = \frac{2a}{a} = 2$$

$$\cot 60° = \frac{OM}{MP} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Now we shall find the value of trigonometrical ratios of angles 0° and 90° also.

Consider the function $\frac{a}{x}$ in which the numerator a has a fixed positive value and the denominator x *varies* through positive values.

For example, when $x = \frac{1}{10}$, $\frac{a}{x} = \frac{a}{\frac{1}{10}}$ *i.e.* $10a$

When $x = \frac{1}{500}$, $\frac{a}{x} = \frac{a}{\frac{1}{500}}$ *i.e.*, $500a$

When $x = \frac{1}{4000}$, $\frac{a}{x} = \frac{a}{\frac{1}{4000}}$ *i.e.*, $4000a$.

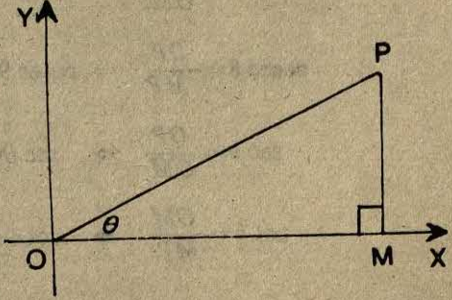
Thus, as x becomes *smaller* and *smaller*, the value of $\frac{a}{x}$ becomes *larger* and *larger.* The value of $\frac{a}{x}$ can be made as large as we please by making x sufficiently small.

Therefore we say that when x tends to zero, through positive values, $\frac{a}{x}$ tends to *infinity*. It is denoted by '∞'.

Note. (1) *Here x is not equal to zero. Division of any number by zero is not defined.*

(2) *Infinity is not a definite number, but a symbol denoting a very very large number.*

To find Trigonometrical ratios of 0°.



Let the revolving line OP, starting from OX, trace out a very small angle $XOP = \theta$, in the positive direction.

From P, draw $PM \perp OX$.

As the angle POM gradually decreases, MP also gradually decreases, *i.e.*, P and M come near and near.

When OP coincides with OM, MP vanishes and P coincides with M.

Then $OP = OM = a$ (say),

$MP = 0$ and $\theta = 0°$.

$$\sin\theta=\frac{MP}{OP} \Rightarrow \sin 0°=\frac{0}{a}=0.$$

$$\cos\theta=\frac{OM}{OP} \Rightarrow \cos 0°=\frac{a}{a}=1.$$

$$\tan\theta=\frac{MP}{OM} \Rightarrow \tan 0°=\frac{0}{a}=0.$$

$$\operatorname{cosec}\theta=\frac{OP}{MP} \Rightarrow \operatorname{cosec} 0°=\frac{a}{\text{very small number}}=\infty \text{ (not defined)}$$

$$\sec\theta=\frac{OP}{OM} \Rightarrow \sec 0°=\frac{a}{a}=1.$$

$$\cot\theta=\frac{OM}{MP} \Rightarrow \cot 0°=\frac{a}{\text{very small number}}=\infty \text{ (not defined)}$$

Note. *The trigonometrical ratios of 0° are really the limiting values of trigonometrical ratios of an angle*

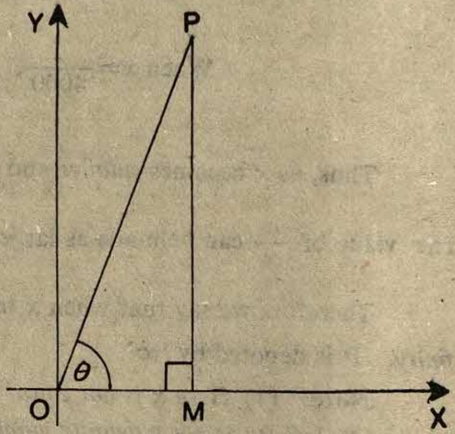
To find the trigonometrical ratios of 90°.

Let the revolving line OP, starting from OX, trace out $\angle XOP=\theta$, a little less than 90°. in the positive direction.

From P, draw $PM \perp OX$.

As the angle POM gradually increases, MP also increases and OM decreases.

When OP coincides with OY, OM vanishes and $\angle POM$ becomes 90°.



Then $OP=MP=a$ (say), $OM=0$ and $\theta=90°$

$$\sin\theta=\frac{MP}{OP} \Rightarrow \sin 90°=\frac{a}{a}=1$$

$$\cos\theta=\frac{OM}{OP} \Rightarrow \cos 90°=\frac{0}{a}=0$$

$$\tan\theta=\frac{MP}{OM} \Rightarrow \tan 90°=\frac{a}{\text{very small number}}=\infty \text{ (not defined)}$$

$$\operatorname{cosec}\theta=\frac{OP}{MP} \Rightarrow \operatorname{cosec} 90°=\frac{a}{a}=1$$

$$\sec\theta=\frac{OP}{OM} \Rightarrow \sec 90°=\frac{a}{\text{very small number}}=\infty \text{ (not defined)}$$

$$\cot\theta=\frac{OM}{MP} \Rightarrow \cot 90°=\frac{0}{a}=0.$$

These values of trigonometrical ratios can be written as under :

Angle	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined
cosec	not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	not defined
cot	not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 1. Find the value of sin 30°+tan 45°+cos 60°.

Solution. $\sin 30^\circ+\tan 45^\circ+\cos 60^\circ=\frac{1}{2}+1+\frac{1}{2}$

$=2$

Example 2. Evaluate, without using tables

(*a*) $\tan^2 30^\circ+\tan^2 45^\circ+\tan^2 60^\circ$ (*b*) $\dfrac{\tan 60^\circ}{\sin 60^\circ-\cos 60^\circ}$

Solution. (*a*) $\tan^2 30^\circ+\tan^2 45^\circ+\tan^2 60^\circ$

$$=\left(\frac{1}{\sqrt{3}}\right)^2+(1)^2+(\sqrt{3})^2$$

$$=\frac{1}{3}+1+3$$

$$=4\frac{1}{3}$$

(*b*) $$\frac{\tan 60^\circ}{\sin 60^\circ-\cos 60^\circ}=\frac{\sqrt{3}}{\frac{\sqrt{3}}{2}-\frac{1}{2}}$$

$$=\frac{\sqrt{3}}{\frac{\sqrt{3}-1}{2}}=\frac{\sqrt{3}}{1}\times\frac{2}{\sqrt{3}-1}$$

$$=\frac{2\sqrt{3}}{\sqrt{3}-1}$$

$$=\frac{2\sqrt{3}\,(\sqrt{3}+1)}{(\sqrt{3}-1)\,(\sqrt{3}+1)}$$ Rationalise the denominator. The rationalising factor is $(\sqrt{3}+1)$.

$$=\frac{2\times 3+2\sqrt{3}}{(\sqrt{3})^2-(1)^2}$$

$$=\frac{6+2\sqrt{3}}{3-1}=\frac{6+2\sqrt{3}}{2}$$

$$=3+\sqrt{3}$$

EXERCISE 9(c)

(Section—A)

1. In answer to each of the following statements, write 'True' or 'False' as appropriate.
   (*a*) $\sin 33°=\cos 60°$.
   (*b*) If $\theta=90°$, then $\cos^2\theta=1$.

2. Given $\sqrt{3}=1{\cdot}732$, express as decimal the following :
   (*i*) $\sin 60°$ (*ii*) $\frac{2}{\tan 30°}$.

3. Angle C is acute and $\sin C=\cos C$. Find the value of C.

(Section—B)

Without using tables, find the values of the following :

4. $\operatorname{cosec} 30°+\cot 45°$.
5. $2\sin 30°\cos 30°$.
6. $\sin^2 30°+\sin^2 45°$.
7. $\cos^2 30°+\cos^2 45°$.
8. $\cot^2 45°+\tan^2 60°$.
9. $\operatorname{cosec}^2 60°-\cot^2 60°$.
10. Prove that $\sin 60°=2\sin 30°\cos 30°$.
11. Verify without using tables that $\cos 60°=\cos^2 30°-\sin^2 30°$.
12. Prove that $\tan 60°=\frac{2\tan 30°}{1-\tan^2 30°}=\sqrt{3}$.
13. If $A=30°$, show that $\cos 2A=1-2\sin^2 A=\frac{1}{2}$.
14. If $A=60°$, $B=30°$, verify that
    (*a*) $\sin(A-B)=\sin A\cos B-\cos A\sin B$.
    (*b*) $\cos(A-B)=\cos A\cos B+\sin A\sin B$.
15. Without using tables find the value of $\frac{\sin^2 45°+\cos^2 45°}{\tan^2 60°}$

Example 1. Find the value of $\sin^2 45°+\sin^2 60°+\sin^2 90°$.

Solution. $\sin^2 45°+\sin^2 60°=\sin^2 90°$

$$=(\sin 45°)^2+(\sin 60°)^2+(\sin 90°)^2$$

$$=\left(\frac{1}{\sqrt{}}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2+(1)^2$$

$$=\frac{1}{2}+\frac{3}{4}+1$$

$$=2\frac{1}{4}$$

Example 2. Find the value of

$4(\sin^4 30^\circ+\cos^4 60^\circ)-3(\cos^2 45^\circ-\sin^2 90^\circ)$ [*C.B.S.E., 1985 (A.I.)*]

Solution. Given Exp.

$$\begin{aligned}
&=4(\sin^4 30^\circ+\cos^4 60^\circ)-3(\cos^2 45^\circ-\sin^2 90^\circ)\\
&=4\left[\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^4\right]-3\left[\left(\frac{1}{\sqrt{2}}\right)^2-(1)^2\right]\\
&=4\left(\frac{1}{16}+\frac{1}{16}\right)-3\left(\frac{1}{2}-1\right)\\
&=4\times\frac{2}{16}-3\left(-\frac{1}{2}\right)\\
&=\frac{1}{2}+\frac{3}{2}\\
&=2
\end{aligned}$$

EXERCISE—9(d)

(Section—A)

1. Calculate the value of each of the following :
   - (*a*) $\sin^2 90^\circ-\tan^2 45'+\sin 45^\circ \cos 45^\circ$.
   - (*b*) $\sin 60^\circ \cos^3 30^\circ+\sin^2 45^\circ \sin 90^\circ$.
   - (*c*) $\sin^4 45^\circ+\tan^2 60^\circ-\text{cosec}^2 90^\circ$.
   - (*d*) $\sec^2 0^\circ-\text{cosec}^2 45^\circ+\cos^2 90^\circ+\cot^2 30^\circ$.
2. Evaluate $\cos 90^\circ+\cos^2 45^\circ \sin 30^\circ \tan 45^\circ$.
3. Prove that $\sin 90^\circ=3 \sin 30^\circ-4 \sin^3 30^\circ$.
4. Prove that $\cos 90^\circ=4 \cos^3 30^\circ-3 \cos 30^\circ$.
5. Find the value of $3 \sin^2 30^\circ+2 \tan^2 60^\circ-5 \cos^2 45^\circ$.
6. Prove that $2 \sin^2 30^\circ-3 \cos^2 45^\circ+\tan^2 60^\circ=2$. [*C.B.S.E., 1986 (A.I.)*]
7. Find the value of $\sin^2 30^\circ-3 \cos^2 45^\circ+4 \tan^2 60^\circ$. [*C.B.S.E., 1987 (A.I.)*]

(Section—B)

8. Prove that

$$3 \tan^2 30^\circ-\frac{4}{3} \sin^2 60^\circ-\frac{1}{2} \text{cosec}^2 45^\circ+\frac{4}{3} \sin^2 90^\circ=\frac{1}{3}.$$

9. Find the value of $\dfrac{4}{\tan^2 60^\circ}+\dfrac{1}{\cos^2 30^\circ}-\sin^2 45^\circ$.

10. Without using tables, find the value of

$$\frac{\sin^2 30^\circ+\cos^2 45^\circ+\sin^2 60^\circ}{\tan^2 30^\circ}$$

11. Find the value of

$\sin^2 30^\circ \cos^2 60^\circ-\cos^2 30^\circ \tan^2 45^\circ$. [*C.B.S.E., 1985 (Delhi)*]

12. Prove that

$$\frac{\cot 45^\circ}{\text{cosec } 30^\circ}-\frac{\sin 30^\circ}{\sin 90^\circ}+\frac{\cos 60^\circ}{\tan 45^\circ}=\frac{1}{2}$$

13. Prove that

$$3 \tan^2 30^\circ+\frac{4}{3} \sin^2 60^\circ-\frac{1}{3} \cos^2 30^\circ-\frac{1}{2} \text{cosec}^2 45^\circ=\frac{3}{4}.$$

14. Simplify $\frac{\cos 45^\circ}{\operatorname{cosec} 30^\circ} - \frac{\sin 30^\circ}{\cot 45^\circ} + \frac{\cos 60^\circ}{\tan 45^\circ}$.

Example 1. Find the value of x if $\sin 2x = \frac{1}{2}$.

Solution. $\sin 2x = \frac{1}{2}$

$$\Leftrightarrow \quad \sin 2x = \sin 30^\circ$$

$$\Rightarrow \quad 2x = 30^\circ$$

$$\therefore \quad x = 15^\circ$$

Example 2. Find x from the equation

$$x \sin 30^\circ \cos^2 45^\circ = \frac{\cot^2 30^\circ \sec 60^\circ}{\operatorname{cosec}^2 45^\circ \operatorname{cosec} 30^\circ}.$$

Solution. $x \sin 30^\circ \cos^2 45^\circ = \frac{\cot^2 30^\circ \sec 60^\circ}{\operatorname{cosec}^2 45^\circ \operatorname{cosec} 30^\circ}$

$$\Leftrightarrow \quad x \times \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{(\sqrt{3})^2 \times 2}{(\sqrt{2})^2 \times 2}$$

$$\Leftrightarrow \quad x \times \frac{1}{2} \times \frac{1}{2} = \frac{3 \times 2}{2 \times 2}$$

$$\Leftrightarrow \quad \frac{x}{4} = \frac{3}{2}$$

$$\Leftrightarrow \quad x = \frac{3}{2} \times 4$$

$$\Rightarrow \quad x = 6.$$

EXERCISE 9(e)

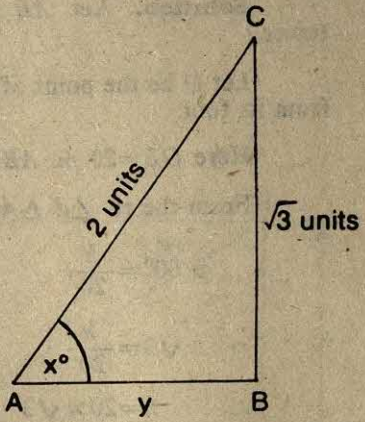
(Section—A)

1. Find the value of x if $\tan 3x = 1$.
2. If $2 \sin 2\theta = \sqrt{3}$, find the value of θ.
3. Find the value of x from $\cos x^\circ = \sin 30^\circ$.
4. If $0 < x < 90^\circ$, find the numerical value of x for which $\tan x^\circ = \cot x^\circ$,
5. If $0 < x < 90^\circ$ give the numerical value of x for which $\sin x^\circ = \cot x^\circ$.

(Section—B)

6. Find the value of x from
$\tan^3 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ \cot 30^\circ$.
7. If $2 \cot 45^\circ - 2 \sin^4 60^\circ + \frac{3}{4} \tan^2 30^\circ = x^2 \cos^3 60^\circ$, find the value of x.

(Section—C)



8. In adjoining figure, ABC is a right angled triangle with B as a right angle.

   Given that $AB=y$ units, $BC=\sqrt{3}$ units, $AC=2$ units and $\angle A=x°$.

   Using trigonometrical ratios find (*a*) x (*b*) $\tan x$ (*c*) use $\cos x$ to find the value of y.

9. A pole 15 *m* long rests against a vertical wall at an angle of 60° with the ground. Calculate :

   (*i*) how high up the wall will the pole reach ?

   (*ii*) how far is the foot of the pole from the wall ?

10. A rhombus of side 10 *cm* has two angles of 60°. Calculate the lengths of the diagonals of the rhombus.

9.6. HEIGHTS AND DISTANCES

We use trigonometry to find heights of objects and distances between points, without actually measuring these heights and distances. In such cases the following two terms are used.

Let OA be a horizontal line in the same vertical plane as an object P.

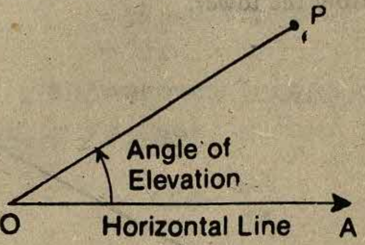


Fig. (*i*)

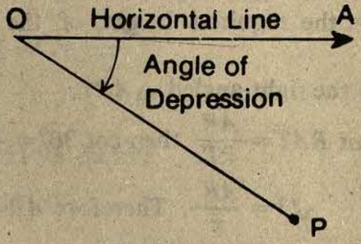


Fig. (*ii*)

If the object is at a *higher level* than our eyes, we have to lift up our head to view the object. In the process, our eyes move through an angle. This angle is called the **angle of elevation of** the object.

In figure (*i*), the object P is above the horizontal line OA. The angle AOP is the **angle of elevation** of the object P as seen from the point O.

If the object is at a *lower level* than our eyes, we have to turn our head downwards to view the object. In the process, our eyes move through an angle. This angle is called the **angle of depression** of the object

In figure (*ii*), the object P is below the horizontal line OA. The angle AOP is called the **angle of depression** of the object P as seen from the point O.

Notice that both angles are measured with the horizontal. Numerically angle of elevation is *equal* to angle of depression.

Example 1. The angle of elevation of the top of a tower at a distance of 20 metres from its foot on a horizontal plane is found to be 60°. Find the height of the tower.

Solution. Let AB represent the vertical height of the tower.

Let O be the point of observation at a distance of 20 metres from its foot.

Here $OB=20\ m$, $AB=h\ m$ and $\angle AOB=60^\circ$.

From the rt. $\angle d$ $\triangle AOB$,

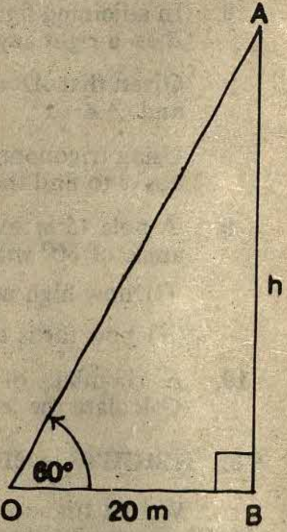
$$\text{n } 60^\circ=\frac{h}{20}$$

$$\Rightarrow \quad \sqrt{3}=\frac{h}{20}$$

$$\therefore \quad h=20\times\sqrt{3}$$

$$=20\times 1{\cdot}732\ m$$

The height of the tower $=$ **33·64 m.**



Example 2. The angular elevation of a tower from a point is 30°, at a point in a horizontal line to the foot of the tower and 100 *m* nearer, it is 60°. Find the height of the tower. Find also the distance of the first point from the tower.

Solution. Let RT be the tower; A and B the points of observation such that

$$\angle RAT=30^\circ,\ \angle RBT=60^\circ$$

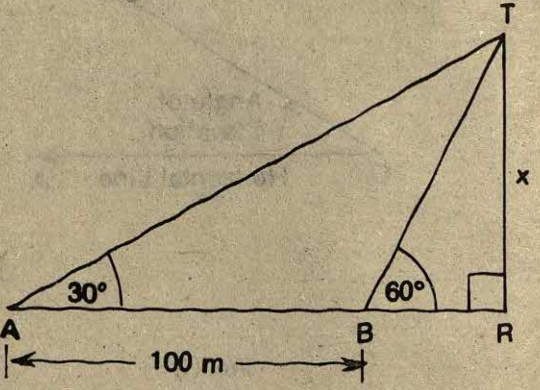
and $\quad AB=100\ m.$

Let the required height of the tower be $x\ m$.

In the right-angled $\triangle ART$,

$$\cot RAT=\frac{AR}{RT} \text{ then } \cot 30^\circ=\frac{AR}{x}$$

$$\therefore \quad \sqrt{3}=\frac{AR}{x}. \text{ Therefore } AR=x\sqrt{3}$$



In the right-angled $\triangle BRT$,

$$\cot RBT=\frac{BR}{RT}. \qquad \text{Then } \cot 60^\circ=\frac{BR}{x}$$

$$\therefore \quad \frac{1}{\sqrt{3}}=\frac{BR}{x}. \qquad \text{Therefore } BR=\frac{x}{\sqrt{3}}$$

But $AB=AR-BR=x\sqrt{3}-\dfrac{x}{\sqrt{3}}$

$$\therefore \quad 100=\frac{3x-x}{\sqrt{3}}$$

or $\quad \dfrac{2x}{\sqrt{3}}=100 \quad$ *i.e.*, $\quad x=50\sqrt{3}$

Hence, the height of the tower $=50\sqrt{3}\ m$

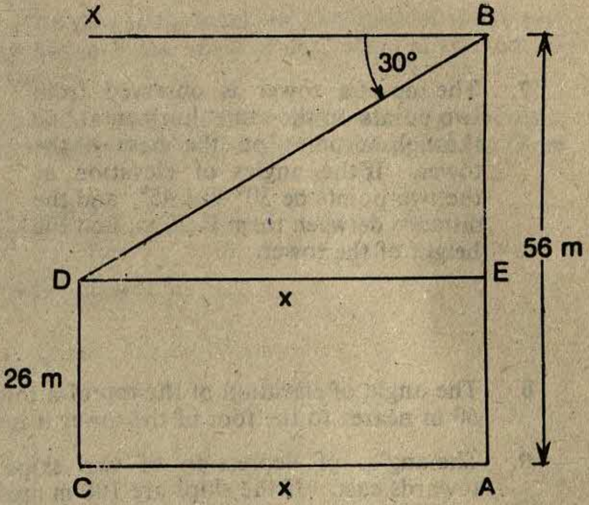
$$=50\times 1{\cdot}732\ m.$$

$$=86{\cdot}6\ m \text{ nearly.}$$

The distance of the first point of observation from the tower

$$= AR = x\sqrt{3} \text{ metres}$$
$$= 50\sqrt{3} \times \sqrt{3} \text{ metres}$$
$$= 150 \text{ metres.}$$

Example 3. The angle of depression of a 26 metres high building from the top of a tower 56 metres high is 30°. Find the distance between the building and the tower.



Solution. Let AB and CD represent the vertical heights of the tower and the building respectively.

Here $AB = 56$ m.

$CD = 26$ m

and $\angle XBD = 30°$

Let AC be x metres.

Then $DE = x$ metres.

$$EB = AB - AE = AB - CD$$
$$= 56 \text{ m} - 26 \text{ m}$$
$$= 30 \text{ m}$$

$\angle BDE =$ alternate $\angle XBD = 30°$.

In the rt. $\angle d\ \triangle BDE$

$$\cot BDE = \frac{DE}{EB} \quad \Rightarrow \quad \cot 30° = \frac{x}{30}$$

$$\Rightarrow \quad x = 30 \cot 30° = 30 \times \sqrt{3}$$
$$= 30 \times 1{\cdot}732 \text{ metres} = 51{\cdot}96 \text{ metres}$$

$\therefore$ The distance between the building and the tower

$= \mathbf{51{\cdot}96}$ **metres.**

EXERCISE 9 (f)

(Section—A)

1. A vertical flagstaff stands on a horizontal plane. From a point distant 30 metres from its foot the angle of elevation of its top is found to be 30° ; find the height of the flagstaff.

2. What is the angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole ?

3. An artist climbs a rope stretched from the top of a pole and fixed on the ground, the height of the pole is 10 metres and the angle made by the rope with the ground is 30°. Find the length of the rope. [*C.B.S.E.*, *1982* (*Delhi*)]

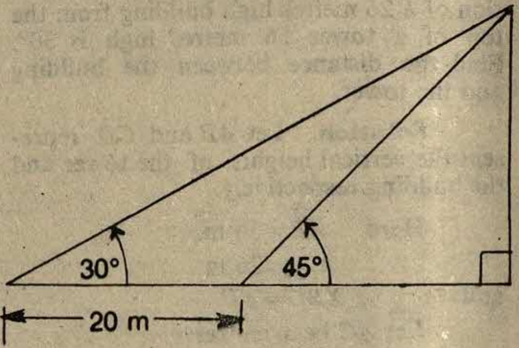
4. The angle of elevation of the top of a tower at a distance of 150 metres from its foot on a horizontal plane is found to be 30°. Find the height of the tower, correct to one place of decimal.

(Section—B)

5. The shadow of a tower standing on a level plane is found to be 60 metres longer when the sun's altitude is 30° than when it is 45°, prove that the height of the tower is $30(\sqrt{3}+1)$ metres. *[C.B.S.E., 1986 (Delhi)]*

6. If the shadow of a pole fixed vertically in earth is 5 m, when the sun's altitude is 30°, what would be the length of the shadow, when the sun's altitude is 60° ?

7. The top of a tower is observed from two points on the same horizontal line through a point on the base of the tower. If the angles of elevation at the two points be 30° and 45°, and the distance detween them is 20 m, find the height of the tower.



8. The angle of elevation of the top of a tower is observed to be 30° and from a point 60 m nearer to the foot of the tower it is 45°. Find the height of the tower.

9. The angles of depression of two ships from the top of a light-house are 45° and 30° towards east. If the ships are 100 m apart, find the height of the light-house.

10. The pilot of a helicopter, at an altitude of 1200 m, finds that two ships are sailing towards it, in the same direction. The angles of depression of the ships as observed from the helicopter are 60° and 45° respectively. Find the distance between the two ships. *[C.B.S.E., 1987 (A.I.)]*

11. From the top of a cliff 200 m high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60°. Find the height of the tower.

12. From the foot of a tower the angle of elevation of the top of a column is 60°, and from the top of the tower, which is 20 m high, the angle of elevation is 30°, find the height of the column.

13. An observer, 1·6 m tall is $30\sqrt{3}$ m away from a tower. The angle of elevation from his eye to the top of the tower is 30°. Determine the height of the tower. *[C.B.S.E., 1986 (A.I.)]*

14. A person standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°. When he retires 10 m from the bank he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

15. The angles of depression of two boats on a river from the top of a tree on the bank of the river are 30° and 60°. If the height of the tree is 30 m and the boats are in line with the tree and on the same side of it, find the distance between the boats.

(Section—C)

16. A ladder of length 4 m makes an angle of 30° with the floor while leaning against one wall of a room. If the foot of the ladder is kept fixed on the floor and it is made to lean against the opposite wall oft he room, it makes an angle of 60° with the floor. Find the distance between these two walls of the room. *[C.B.S.E., 1983 (A.I.)]*

17. Two pillars of equal height stand on either side of a roadway which is 100 metres wide ; at a point in the roadway between the pillars the elevation of the top of the pillars are 60° and 30°. Find their height and the position of the point.

18. Two men are on the opposite sides of a tower. They measure the angles of elevation of the top of the tower 45° and 60° respectively. If the height of the tower is 40 metres, find the distance between the two men.

19. A tower subtends an angle α at a point P on the same level as the foot of the tower, and at a second point Q, h vertically above P, the angle of depression of the foot of the tower is β. Find the height of the tower.

20. Two stations due south of a tower which leans towards the north, are at distances a and b from its foot ; if α and β be the elevations of the top of the tower from these stations, prove that

$$\cot\theta=\left(\frac{b\cot\alpha-a\cot\beta}{b-a}\right),$$

where θ is the inclination of the tower to the horizontal.

REVIEW EXERCISE IX

(Section—A)

1. State for each of the following statements whether it is True or False :

(*a*) In a triangle, right-angled at B, the ratio $AB : BC$ is same as the ratio $\sin A : \cos A$.

(*b*) $\cos 30°$ is equal to $\frac{1}{2}$.

(*c*) If $\sin x = \cos y$, then $x + y = 45°$.

(*d*) The angle of elevation of B from A is equal to the angle of depression of A from B.

2. Fill in the blanks to make each of the following statements true :

(*a*) $\sec^2 60° - \tan^2 60° = \ldots\ldots$ [*C.B.S.E., 1987 (A.I.)*]

(*b*) The minimum value of $\sin A$, $(0° \leqslant A \leqslant 90°)$ is...... [*C.B.S.E., 1986 (A.I.)*]

(*c*) The value of $\sin^2 60° + \sin^2 30°$ is...... [*C.B.S.E., 1986 (Delhi)*]

(*d*) The measure of the acute angle θ for which

$$\sin \theta = \cos \theta = \frac{1}{\sqrt{2}} \tan \theta \text{ is......}$$

[*C.B.S.E., 1985 (A.I.)*]

(*e*) The maximum value of $\sin A$, $(0° \leqslant A \leqslant 90°)$ is...... [*C.B.S.E., 1985 (Delhi)*]

(*f*) If $\cos \theta - \sin \theta = 0$, then the value of θ being in the first quadrant is...... [*C.B.S.E., 1982 (A.I.)*]

3. For any angle θ, state the value of $\sin^2 \theta + \cos^2 \theta$.

4. If $0 \leqslant x \leqslant 90°$. state the numerical value of x for which $\sin x^0 = \cos x^0$.

5. Given $\tan \theta + \cot \theta = 2$, find the value of $\tan^2 \theta + \cot^2 \theta$.

6. Prove that $\cos 60° = 2 \cos^2 30° - 1$. (Do **not** use tables)

7. **Without** using tables find the value of

$$\cos^2 45° + \sin^2 60° + \sin^2 30°.$$

(Section—B)

8. **Without** using tables find the value of

$$\frac{\sin 30° - \sin 90° + 2 \cos 0°}{\tan 30° \times \tan 60°}$$

9. Find the value of

$$\frac{\tan 60° - \tan 30°}{1 + \tan 60° \tan 30°}$$

[*C.B.S.E., 1986 (A.I.)*]

10. In a right-angled triangle, A is an acute angle and $\sin A = \frac{3}{\sqrt{18}}$.

Without using tables, find the value of (*i*) $\sec A$ and (*ii*) $\cot A$.

11. If $\sin \theta = \frac{6}{10}$, find without using tables, the value of $\cos \theta + \tan \theta$.

12. If $\cos \theta = \frac{3}{5}$, find the value of $\dfrac{\sin \theta - \dfrac{1}{\tan \theta}}{2 \tan \theta}$.

(Section—C)

13. Two persons standing on the same side of a tower, measure the angles of elevation of the top of the tower as 30° and 60°. If the height of the tower is 40 metres, find the distance between the two persons.

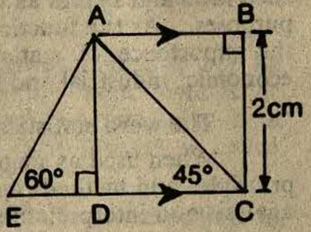
14. Two men are on the opposite sides of a tower. They measure the angles of elevation of the top of the tower as 30° and 60° respectively. If the height of the tower is $50\sqrt{3}$ m, find the distance between the men. [*C.B.S.E., 1987* (*Delhi*)]

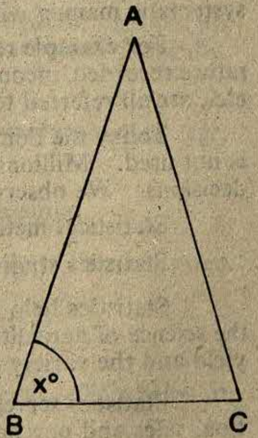
15. The angles of depression of a point on the level ground as viewed from a 20 m high window and the top of the building are 30° and 45°. Calculate the height of the building. [*C.B.S.E., 1986* (*Delhi*)]

16. The length of a string between a kite and a point, on the roof of a building 10 m high, is 180 m. If the string makes an angle θ with level ground such that $\tan\theta=\frac{4}{3}$, how high is the kite from the ground? (Assume there is no slack in the string). [*C.B.S.E., 1986* (*A.I.*)]

17. In the adjoining figure, *AB* and *EC* are parallel to each other. Sides *AD* and *BC* are 2 cm each and are perpendicular to *AB*. Given that $\angle AED=60°$ and $\angle ACD=45°$, without using tables, calculate

(*i*) *AB*, (*ii*) *AC* and (*iii*) *AE*.





In the given diagram, *ABC* is an isosceles triangle in which it is given that

$$AB=AC=10 \text{ cm},\ BC=6 \text{ cm and } \angle B=x^0$$

(*a*) Write down the value of $\cos x^0$.

(*b*) If *CM* is drawn perpendicular to *AB*, write down an expression for *BM* in terms of *x* and hence find the length of *BM*.

□ □

10

STATISTICS

10.1. STATISTICS

The word '*statistics*' has been derived from the Latin word 'statis' which means a political state. In the early days, the study of statistics was confined only to the collection of such facts and figures as were required by the state for taxation, land-distribution and military purposes. As the functions of the state increased the scope of statistics became wider. Today the importance of statistics is growing practically in all spheres of life—political, social economic, industrial and scientific.

The word statistics is used in *two* different senses—one singular and the other plural.

When used as singular, statistics refers to the subject as a whole. Then it denotes the principles and methods used in collection of data, its classification and tabulation, and its analysis and interpretation.

When used as plural, statistics refers to the data (numerical facts) itself, collected in a systematic manner with some definite purpose in view, in any field of enquiry.

For example records of height and weight of students of a school, rainfall and temperature recorded, income and expenditure of a government, imports and exports of a country, etc., are all referred to as statistics.

Today the domain of statistics is very wide. There is hardly any field in which statistics is not used. Millions of people all over the world use statistical methods in their day-to-day decisions. We observe that human behaviour and statistical methods have much in common.

Statistical methods make huge data intelligible and readily understandable.

Statistics studies relationship between different phenomena *e.g.*, demand and price.

Statistics help in solving various difficulties or problems of other sciences. For example, the science of agriculture uses statistics in carrying out experiments for the improvements of the yield and the variety of crops.

Statistics serves as a guide for planning. It helps a businessman to regulate his production, sales and purchases. State budgets are prepared with the help of statistics.

Bankers, stock exchange brokers, investors, etc., make extensive use of statistical data. Insurance companies cannot carry on their business without statistical data relating to life tables and premium rates, etc.

Statistics helps states to run the administration.

10.2. COLLECTION OF DATA

The first step in every statistical investigation is the collection of facts and figures or data. Recording of age of each student in your class is collection of data of a particular type.

Before collecting any data, you must clearly understand the purpose of collection of statistical data, the scope and its limitations.

There are two types of data :

(*i*) **Primary data** (*ii*) **Secondary data**

Primary data are the data collected for the first time by the investigator himself or enumerators working under him.

The methods used to collect primary data are called *primary methods.*

Suppose you are finding the height of each student in your class. A record of this information made by you is primary data.

Secondary data are the data collected by some person or organisation other than the investigator for their own use.

The methods used to get secondary data are called *secondary methods.*

10.3. CLASSIFICATION

The data collected by primary methods are usually very large. As such it will not be easy to find the information contained in the data or discover easily the main features or characteristic. So the unimportant details and figures are discarded and the remaining data consisting of the significant figures are compiled in a prominent form. Such process is called **classification**.

Classification is a process of arranging the data into groups or classes according to their resemblance with respect to some variable character.

Classification brings like things at one place and unlike at a different place. It presents the data in a systematic manner. It helps in understanding the data without much difficulty. It makes the task of comparison easy.

Any character which can vary in quality or magnitude is called a **variate** ; thus age, height, occupation, income, colour of hair, examination marks are variates.

Age, height and income are **quantitative** variates as they can be measured.

Colour of hair, and occupation are **qualitative** variates as they cannot be measured numerically.

10.4. TABULATION

After the data have been classified, they may be put into a tabular form. This helps in understanding of the data and its proper interpretation.

Tabulation is the process of systematic presentation of the numerical data in an orderly manner into rows and columns.

The use of columns and rows for entering figures facilitates comparison.

10.5. FREQUENCY DISTRIBUTION

An arrangement of data in a systematic order is called a *distribution* or a *series.*

There are two types of series (*i*) *discrete*, (*ii*) *continuous.*

A variable which takes only integral values is called a *discrete series.*

Examples of discrete variable are number of pupils in a class, number of workers in a factory, etc.

A variable which can take any possible value between certain limits is called a *continuous variable.*

Examples of continuous variable are age, weight, height of pupils in a class.

Marks obtained by 40 students of IX class in Mathematics in the half-yearly examination are given below :

12	18	14	17	19	18	28	15	24	18
15	38	24	20	16	20	30	19	26	17
34	22	15	18	17	30	16	17	20	17
26	19	28	24	20	30	34	28	21	30

The data given above is called *ungrouped data.*

Let us arrange the data in *ascending order of magnitude.*

12, 14, 15, 15, 15, 16, 16, 17, 17, 17, 17, 17, 18, 18,
18, 18, 19, 19, 19, 20, 20, 20, 20, 21, 22, 24, 24, 24,
26, 26, 28, 28, 28, 30, 30, 30, 30, 34, 34, 38.

We observe that there is 1 student who gets 12 marks. There are 3 students who get 15 marks, etc.

Now the data can be written in tabular form like this :

Marks	*Number of Students*	*Marks*	*Number of Students*
12	1	21	1
14	1	22	1
15	3	24	3
16	2	26	2
17	5	28	3
18	4	30	4
19	3	34	2
20	4	38	1

This is called a *discrete series*. It is an improvement over the arrangement of figures. It presents a clear picture of data.

These marks can be further classified into groups of classes. Here lowest marks are 12 and highest 38. So, we can consider the following groups of marks :

10—15, 15—20, 20—25, 25—30, 30—35 and 35—40. These groups are called *class-interval* groups. For class-interval 10—15, 10 is the *lower limit* and 15 is the *upper limit*. 15—10 *i.e.*, 5 is the *width* of the class-interval.

We see that 1+1 *i.e.*, 2 students are getting marks between 10 and 15. The class-interval 10—15 denotes all values from 10 but below 15. Note that 15 will be counted in the class-interval 15—20 and not in 10—15

3+2+5+4+3 *i.e.*, 17 students are getting marks between 15 and 20 and so on.

Thus 2 is the *frequency* for the class-interval 10—15 and 17 is the frequency for the class 15—20. The frequency is represented by *f*.

Class-intervals (*Marks*)	*Frequency* (*number of students obtaining these marks*) *f*
10—15	2
15—20	17
20—25	9
25—30	5
30—35	6
35—40	1
Total	40

This is called a *continuous series*.

You should remember the following points while forming a frequency table :

(1) The size or width of class-intervals should be equal.

(2) The class-limits should be whole numbers.

(3) The number of class-intervals should not be ordinarily less than 10 or more than 15.

In the above frequency distribution, the upper limit of one class is the lower limit of the *next* class. It is known as *exclusive method*. This method ensures continuity of data.

When these limits of each class-interval are different, it is known as *inclusive method.* In this method the upper limit of one class is included in that class itself. So the data given on page 152 can be put in the following table :

Class-intervals (*Marks*)	*Frequency* (*Number of students*)
11—15	5
16—20	18
21—25	5
26—30	9
31—35	2
36—40	1
Total	40

Sometimes a column of *cumulative* frequency is added in the simple frequency table. The cumulative frequcncy of a class is obtained by adding together the frequencies of the classes upto that class including frequency of that particular class.

Let us now make the cumulative frequency table for marks obtained by 40 students given on page 153.

Marks	*Number of students* (*f*)	*Cumulative frequency* (*cf*)		
10—15	2			2
15—20	17	2+17	*i.e.,*	19
20—25	9	19+9	*i e.,*	28
25—30	5	28+5	*i.e.,*	33
30—35	6	33+6	*i.e.,*	39
35—40	1	39+1	*i.e.,*	40

This is called a **cumulative frequency table.**

In the first class-interval lower limit is 10 and upper limit is 15 The difference between the upper limit and the lower limit of a class gives the *size* of the class-interval. Here 15—10 *i.e.,* 5 is the size of this class-interval.

The mid-value in a particular class is called the *class mark* of that class.

Class mark of the above class$=\dfrac{10+15}{2}$ *i.e.,* 12·5 *i.e.*, class mark of a class $= \dfrac{\text{lower class limit}+\text{upper class limit}}{2}$.

Thus, the class marks of the above distribution are 12·5, 17·5, 22·5, 27·5, 32·5 and 37·5.

Example 1. The class marks of a distribution are 28, 34, 40, 46 and 52. Determine the class size and the class limit.

Solution. The class size is the difference between the class marks of two adjacent classes.

$$\therefore \qquad \text{class size} = 34 - 28 = 6$$

Now 28 is the mid point of a class whose width in 6. Therefore, the first class interval is $28 - \frac{6}{2}$, $28 + \frac{6}{2}$ *i.e.* 25, 31 or 25–31.

Similarly next class interval is $34 - \frac{6}{2}$, $34 + \frac{6}{2}$ *i.e.*, 31, 37 or 31–37.

Hence the classes are 25–31, 31–37, 37–43, 43–49 and 49–55.

Thus, we get the following class limits :

25, 31 ; 31, 37 ; 37, 43 ; 43, 49 ; 49, 55.

We form the frequency distribution in the following steps :

(1) *We determine the minimum and maximum values of the variate occurring in the data.*

(2) *We decide upon the number of classes to be formed. We usually keep the number of classes from 5 to 15.*

(3) *The difference between maximum value and minimum value in a data is called range.*

(4) *We divide range by the number of classes to be formed. Such a quotient gives us the approximate size of a class. We choose a convenient whole number about the quotient.*

(5) *We must have classes to include minimum and maximum numbers occurring in the data.*

(6) *We consider each item from the data, one at a time and put a tally mark (1) against the class to which the item belongs. We record tally marks in bunches of five. The fifth one is marked as a crossing diagonally the other four like* 卌.

(7) *We count the total number of tally marks in each class which gives the frequency of the class.*

The following illustration will further clarify the procedure.

Example 2. The weights in grams of 50 apples picked at random from an orchard are as follows :

50, 60, 55, 115, 80, 105. 45, 65, 70, 85,
105, 80, 75, 65, 75, 100, 90, 85, 55, 50,
65, 55, 45, 70, 60, 50, 45, 95, 90, 80,
90, 80, 75, 115, 105, 85, 75, 60, 55, 75,
125, 140, 150, 165, 130, 155, 160. 95, 100, 135.

Construct a cumulative frequency table. if one class is 80–100 (100 not included).

Solution Here, the minimum observation is 45 and maximum is 165.

One of the class interval is 80–100, other classes of equal sizes covering the data are 40–60, 60–80, 80–100, 100–120, 120–140, 140–160, 160–180.

Cumulative Frequency Table

Class interval (weights in grams)	*Tally Marks*	*Frequency (Number of apples)*	*Cumulative Frequency*
40—60	𝍸 𝍸	10	10
60—80	𝍸 𝍸 |||	13	10+13 *i.e.*, 23
80—100	𝍸 𝍸 ||	12	23+12 *i.e*, 35
100—120	𝍸 ||	7	35+7 *i.e.*, 42
120—140	|||	3	42+3 *i.e.*, 45
140—160	|||	3	45+3 *i.e.*, 48
160—180	||	2	48+2 *i.e.*, 50
Total		50	

How many apples of weight less than 60 grams were picked ?

How many apples of weight less than 80 grams were picked ?

How many apples of weight less than 100 grams were picked ?

Now the above cumulative frequency table can also be expressed in one of the following two ways :

Weights in grams	*Number of apples*
Less than 40	0
Less than 60	10
Less than 80	23
Less than 100	35
Less than 120	42
Less than 140	45
Less than 160	48
Less than 180	50

Weights in grams	*Number of apples*
More than 180	0
More than 160	2
More than 140	5
More than 120	8
More than 100	15
More than 80	27
More than 60	40
More than 40	50

Example 3. Given below is a cumulative frequency distribution table showing the number of packages and the number of times a given number of packages were received by a post office in 60 days :

Number of packages	*Number of times received in 60 days*
Below 10	17
Below 20	22
Below 30	29
Below 40	37
Below 50	50
Below 60	60

Construct a frequency table from the above data.

Solution :

Number of packages	*Frequency*
0—10	17
10—20	22—17 *i.e.*, 5
20—30	29—22 *i.e.*, 7
30—40	37—29 *i.e.*, 8
40—50	50—37 *i.e.*, 13
50—60	60—50*i .e.*, 10
Total	60

EXERCISE 10 (a)

(Section—A)

1. What are the two senses in which the word 'statistics' is used ?
2. What is statistics ? Give important uses of statistics.
3. Distinguish between primary data and secondary data.
4. What do you understand by the classification of data ?
5. What do you mean by tabulation of data ?
6. What are discrete and continuous series ? Give examples.

7. Explain the following with suitable examples :
   (*a*) Class-interval
   (*b*) Width of the class interval
   (*c*) Simple frequency table
   (*d*) Cumulative frequency table
   (*e*) Class mark.

(Section—B)

8. The marks of the candidates in the subject Mathematics are given below :

45 48 15 23 30 35 40 11
29 0 3 12 48 50 18 30
15 30 11 42 23 3 2 44

The maximum marks are 50. Make a frequency distribution taking class intervals 0—10, 10—20, —

9. The following are the monthly rents (in rupees) of 40 shops :

38, 42, 49, 37, 82, 37, 75, 62, 54, 79,
84, 75, 63, 44, 74, 44, 36, 69, 54, 48,
74, 39, 48, 45, 61, 71, 47, 38, 80, 51,
31, 38, 43, 75, 67, 77, 47, 64, 84, 81,

Using the class-intervals 30—38 (excluding 38), 38—46 (excluding 46), etc., construct a frequency table for the above data.

10. Given below are the marks obtained by 40 students in an examination :

29, 45, 23, 30, 40, 11, 48, 01, 15, 34,
40, 03, 12, 48, 49, 18, 30, 24, 25, 29,
31, 32, 25, 22, 27, 41, 12, 13, 02, 44,
07, 43, 09, 49, 19, 13, 32, 39, 25, 03.

Taking class-intervals 1—10, 11—20,..........., 41—50, make a frequency table for the above distribution.

11. The number of match sticks in 40 match boxes, on counting was found as given bellow :

44, 41, 42, 43, 47, 50, 51, 49, 43, 42,
40, 42, 44, 45, 49, 42, 46, 49, 45, 49,
45, 47, 48, 43, 43, 44, 48, 43, 46, 50,
43, 52, 46, 49, 52, 51, 47, 43, 43, 45.

Taking classes 40—41, 42—43, etc., construct the frequency distribution table for the above data.

12. Present the following data of the percentage marks of 60 students in the form of frequency table with 10 classes of equal width, one class being 40—49 :

41, 17, 83, 63, 54, 92, 60, 58, 70, 06,
67, 82, 33, 44, 57, 49, 34, 73, 54, 63,
36, 52, 32, 75, 60, 38, 09, 79, 28, 30,
42, 93, 43, 80, 03, 32, 57, 67, 24, 64,
63, 11, 35, 82, 10, 23, 0, 41, 60, 32,
72, 53, 92, 88, 62, 55, 60, 33, 40, 57.

13. The daily earnings of thirty drug stores are given below :

Daily earnings (in rupees)	*Number of stores*
0—50	3
51—100	7
101—150	4
151—200	5
201—250	4
251—300	3
301—350	2
351—400	2

Construct a cumulative frequency table for the above data.

14. From the ordinary frequency table from the cumulative frequency table given below :

Class-interval	*Cumulative frequency*
0—8	10
8—16	17
16—24	33
24—32	57
32—40	72
40—48	79

(Section—C)

15. Present the following as an ordinary grouped frequency table :

Marks	*Number of students*
Below 10	5
Below 20	12
Below 30	32
Below 40	40
Below 50	45
Below 60	48

[*C.B.S.E., 1984 (A.I.)*]

16. Form a frequency distribution table for the following data :

Marks	*Number of students*
Above 0	30
Above 10	26
Above 20	21
Above 30	14
Above 40	10
Above 50	0

[*C.B.S.E., 1981 (A.I.)*]

10.6. STATISTICAL GRAPHS

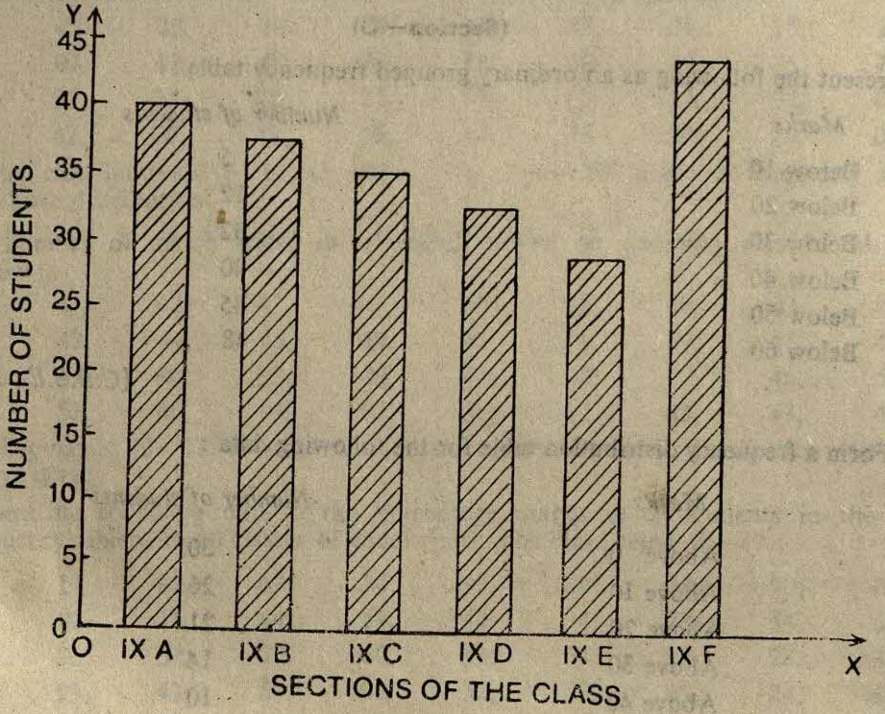
Most people in reading a scientific article or a daily newspaper either skip over the columns of figures and tables altogether or at the most get only a hazy idea about them. Moreover, these figures do not have a lasting effect on their minds. But when these figures are given in form of pictures, the readers quickly notice many facts that could have been found from the figures after studying them. A picture depicting a given data, is called a **statistical graph**. The data presented in the form of graphs is more appealing to the eye and looks attractive. They also help us in comparisons of different data. However, one disadvantage with the graphs is that they do not give *accurate* measurements.

10.7. BAR GRAPHS

The simplest and the most widely used graph is the *bar graph*. With its help, we compare simple magnitudes. The bar graph shown below represents the number of students in different sections of class IX in a school in New Delhi as given in the table :

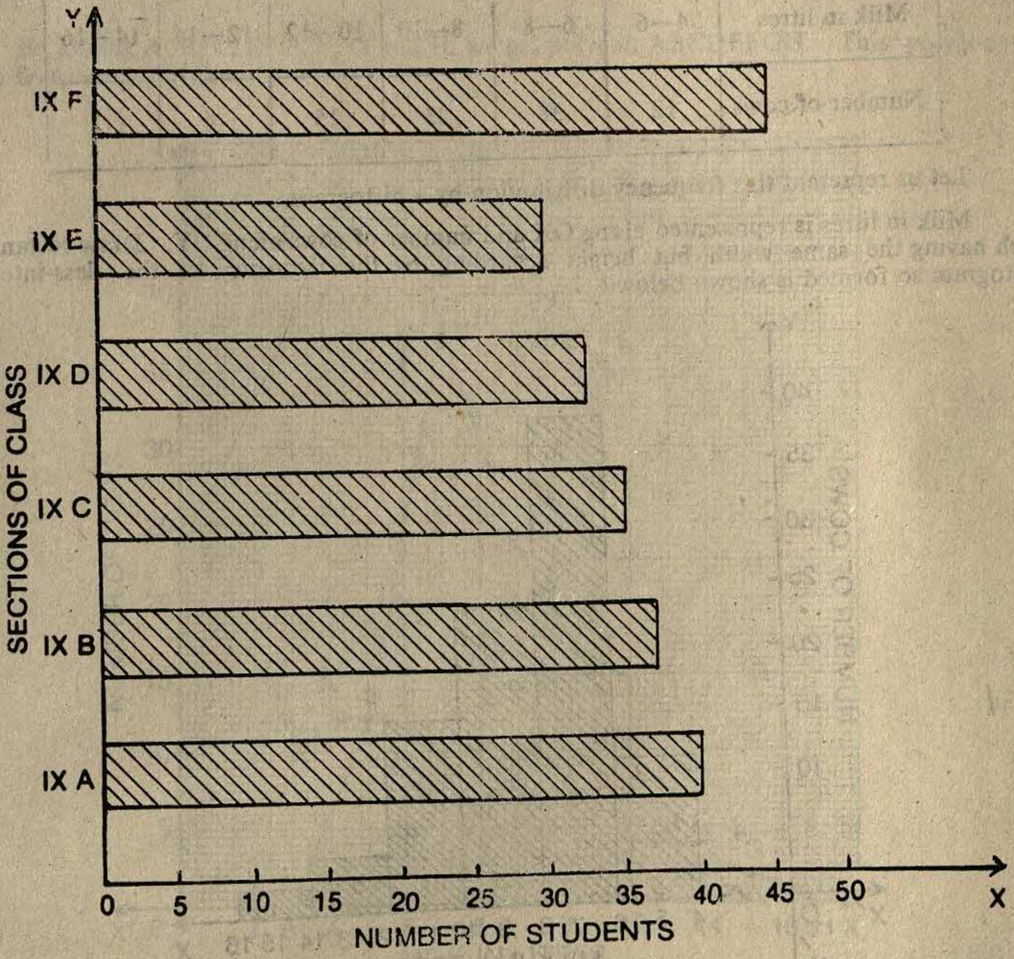
Class	IX A	IX B	IX C	IX D	IX E	IX F
Number of Students	40	37	35	32	29	44

We draw bars of width 1 cm each. The bars are drawn at *equal* distances. The height of each bar is taken according to the strength of each respective section. Taking 8 mm to represent 5 students, we have drawn bar chart as shown below :



Instead of a vertical bar-graph, we can draw horizontal bar-graph for the above data.

Here 1 cm represents 5 students along OX.



10.8. GRAPHS FOR FREQUENCY DISTRIBUTIONS

In case of grouped data, we use other types of graphs. These include (1) Histograms or Block Diagrams, (2) Frequency Polygon and Frequency Curve, (3) Cumulative Frequency Curve, etc.

In these graphs, the classes are generally taken along the X-axis and frequencies along the Y-axis.

10.9. HISTOGRAMS

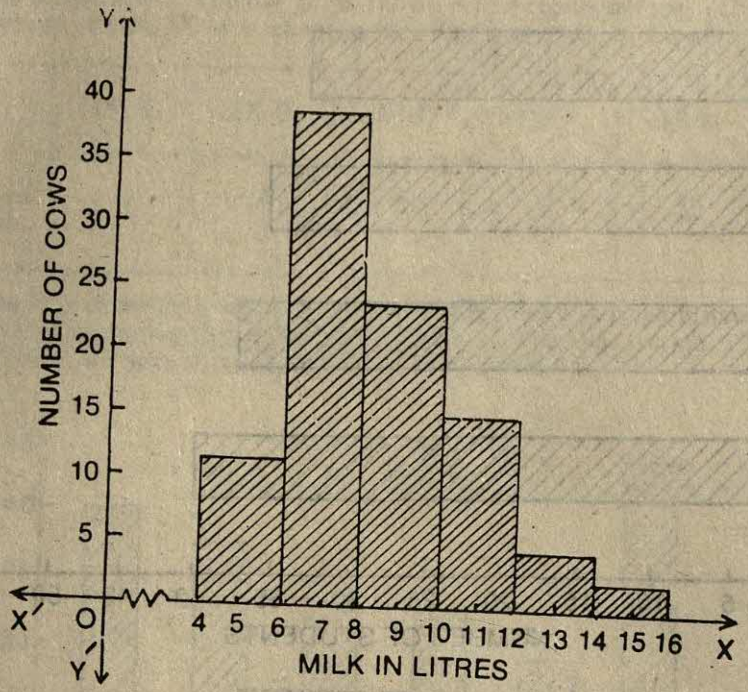
In this type of graph rectangles are constructed with class-interval and frequency as its two adjacent sides. The class-intervals are taken on the X-axis and the frequencies along the Y-axis. As the class intervals are equal, all rectangles will have the same bases but different heights. These diagrams are very appealing to the eye and give an idea of frequency distribution quickly.

94 cows have been classified on the basis of the amount of milk given by them daily.

Milk in litres	4—6	6—8	8—10	10—12	12—14	14—16
Number of cows	12	38	23	15	4	2

Let us represent this frequency distribution by a histogram.

Milk in litres is represented along OX and number of cows along OY. Draw rectangles, each having the same width but height according to the frequency of the class-interval. Histogram so formed is shown below :



Since the scale on the x-axis starts at 4, a *break* is indicated near the origin. It shows that the graph is drawn to scale beginning at 4 and not at the origin.

We draw histograms in the following steps :

(1) *We represent class limits along x-axis, taking a suitable scale.*

(2) *We mark frequencies along y-axis, taking a suitable scale.*

(3) *We draw rectangles with bases along x-axis and height along y-axis.*

Note that it not necessary to take the *same scale* for both the axes.

In the above example, the class intervals are continuous. If the class intervals are not continuous in the given distribution, we must convert the given class intervals into continuous intervals. We should, then, draw the histogram with reference to these continuous class intervals.

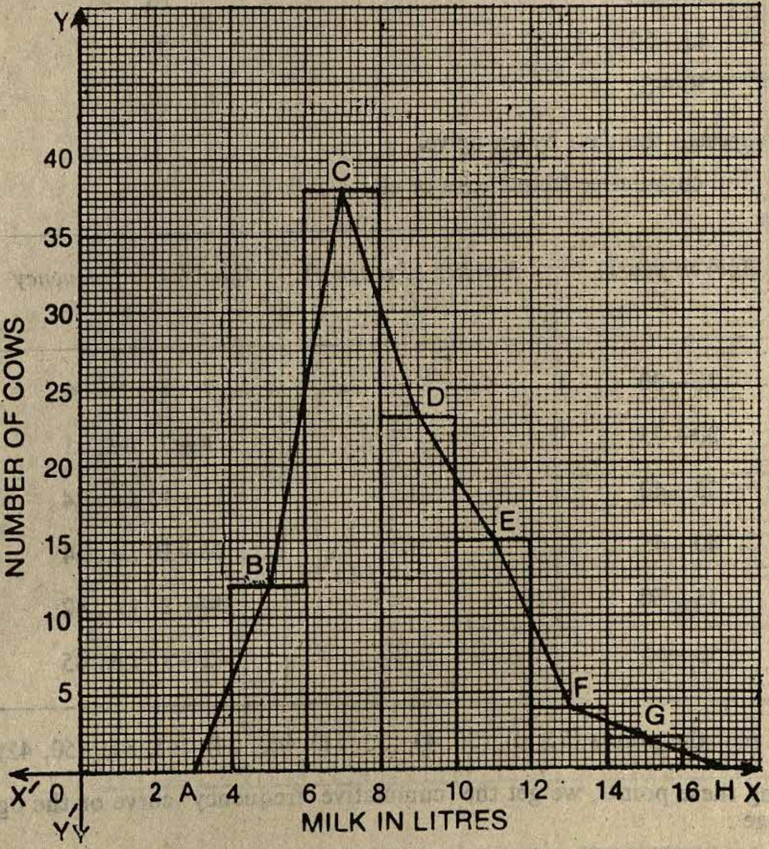
10.10. FREQUENCY POLYGON

In a frequency polygon the variables are assumed to be concentrated at the mid-points of the class-intervals. We make histogram for the frequency distribution and find the mid-point of the upper-side of each rectangle. Then we join these mid-points.

Let B, C, D, E, F and G be the mid-points of the rectangles.

The mid-points at each end are joined to mid-points, at zero frequency of the adjacent classes.

By joining A, B, C, D, E, F, G and H, we get polygon ABCDEFGH. This polygon is called **frequency polygon.**



If instead of joining the points plotted for the frequency polygon by segments they are joined by a smooth curve, a **frequency curve** will be obtained. It helps us in smoothening the change.

Since the frequency curve is a *smooth* drawing and no break or ups and downs are permissible in drawing such curves.

10.11. CUMULATIVE FREQUENCY CURVE

In a frequency curve, we represent mid-points of class-intervals along x-axis and frequencies along y-axis.

Now we will represent upper limits of classes along x-axis and cumulative frequencies of respective classes along y-axis. The curve obtained by joining the resulting points is called **the cumulative frequency curve** or the **ogive**. So an ogive is the graphical representation of a cumulative frequency distribution.

Consider the following distribution showing ages in years of 45 teachers in our school.

Ages in years	Number of teachers
25—30	4
30—35	7
35—40	13
40—45	10
45—50	8
50—55	3

Let us represent the data by an ogive.

First we find cumulative frequencies of classes.

Ages in years	*Number of teachers* f	*Cumulative frequency*
25—30	4	4
30—35	7	4+7 *i.e.*, 11
35—40	13	11+13 *i.e.*, 24
40—45	10	24+10 *i.e.*, 34
45—50	8	34+8 *i.e.*, 42
50—55	3	42+3 *i.e.*, 45

Now we plot the points (25, 0), (30, 4), (35, 11), (40, 24), (45, 34), (50, 42) and (55, 45).

By joining these points, we get the cumulative frequency curve or the ogive as shown on the next page :

We draw an ogive for a given frequency distribution in the following steps :

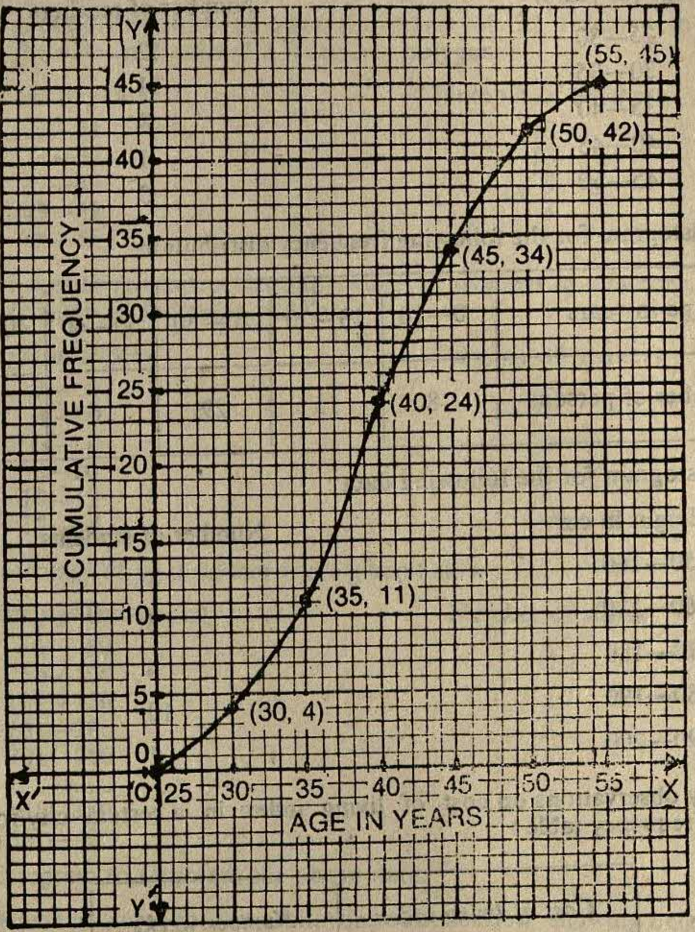
(1) *We form the cumulative frequency table.*

(2) *We mark upper class limits along x-axis.*

(3) *We mark cumulative frequencies along y-axis.*

(4) *We plot the points taking upper limit of each class as the abscissa and its corresponding cumulative frequency as the ordinate.*

(5) *We join these points by a* free-hand curve.



Note that an ogive starts from a point on the x-axis representing the lower limit of the first class interval.

EXERCISE 10(b)

(Section—A)

1. The following table shows how 59 employees of our school get to school :

Mode of transport	Number of employees
Cars	4
Scooters	9
Bicycles	5
D.T.C. Buses	12
School Buses	16
On foot	3

Draw a bar-chart for the above information.

2. Represent the following distribution of ages (in years) of 35 teachers in a school by means of histogram :

Age (in years)	Number of teachers
25—30	12
30—35	11
35—40	8
40—45	1
45—50	3

3. Draw a histogram for the following frequency distribution :

Age in years	0—10	10—20	20—30	30—40	40—50	50—60
Number of persons	8	15	29	16	7	3

4. Draw a histogram for the following data :

Height in cm	Number of students
150—160	8
160—170	3
170—180	4
180—190	10
190—200	2

5. The weekly observations on cost of living index in a certain city for a year give the following frequency table :

Cost of living index	*Number weeks*
140—150	5
150—160	10
160—170	16
170—180	12
180—190	7
190—200	2
Total	52

Draw a histogram to represent the data.

[C.B.S.E., 1978 (Delhi)]

(Section—B)

6. Following is the distribution of ages (in years) of 32 teachers in a primary school. Represent the data by means of a frequency polygon.

Ages in years	Number of teachers
25—31	10
31—37	13
37—43	5
43—49	3
49—55	1

7. The following data represents the daily wages of workers in a factory. Draw a suitable frequency polygon to represent the data :

Wages in rupees	Number of workers
2— 4	16
4— 6	11
6— 8	20
8—10	25
10—12	18
12—14	7

8. The daily earnings of 30 drug stores are given below :

Daily earnings (in rupees)	Number of stores
0— 50	3
50—100	7
101—150	4
151—200	5
201—250	4
251—300	3
301—350	2
351—400	2

Draw a histogram to represent the above data.

9. Draw a frequency curve for the following distribution showing height of 50 pupils in class I in our school :

Height in cm	100—110	110—120	120—130	130—140	140—150
Number of pupils	8	10	7	15	10

(Section—C)

10. The daily wages of casual labour employed by a factory are given below :

Daily wages in rupees	Frequency
3— 5	8
5— 7	11
7— 9	25
9—11	50
11—13	7
13—15	4

Draw an ogive to represent the above data.

11. The following table gives the distribution of students according to marks obtained by them.

Marks	0—10	10—20	20—30	30—40	40—50
Frequency	8	8	23	12	2

Draw the cumulative frequency curve for the above data.

12. Draw a cumulative frequency curve (an ogive) for the following data :

Age (in years)	Number of patients
0—10	15
10—20	12
20—30	8
30—40	20
40—50	27

13. The data given below represents the marks obtained by 35 students.

21	26	21	20	23	24	22	19	24
26	25	23	26	29	21	24	19	25
26	25	22	25	23	27	26	24	25
30	25	23	27	28	24	28	28	

Taking class-intervals 19—20, 21—22, etc., make a frequency distribution for the above data.

Then construct a combined histogram and frequency polygon for the above data.

14. The data below gives the earning of 350 workers in a cotton mill.

(*a*) Draw the frequency polygon representing the data.

(*b*) Draw also the ogive for the data.

Monthly Earnings in Rupees	Number of workers
160—180	40
180—200	54
200—220	60
220—240	72
240—260	45
260—280	32
280—300	28
300—320	15
320—340	4

[*C.B.S.E., 1985* (*A.I.*)]

REVIEW EXERCISE X

(Section—A)

1. Fill in the blanks to make the following statements true :

(*a*) The difference between the maximum and the minimum observations in a data is called its .... [*C.B.S.E., 1984* (*A.I.*)]

(*b*) If the class marks of a distribution are 32, 39, 46, 53, then the width of each class-interval is .... [*C.B.S.E., 1983* (*A.I.*)]

(*c*) The class mark of the class-interval 8·5—15·5 is .... [*C.B.S.E., 1983* (*Delhi*)]

(*d*) The range of the data 41, 68, 7, 53, 4, 75, 11, 89, 47 is .... [*C.B.S.E., 1982* (*A.I.*)]

(*e*) The number of observations in a class-interval is called the.........of the class-interval. [*C.B.S.E., 1982* (*Delhi*)]

(*f*) The mid-point of a class-interval is called its ....

(*g*) Data are said to be ...., if the investigator himself is responsible for the collection of data.

2. Following are the data on the daily wages of casual labour employed by a group of limited concerns.

Daily wages in (*rupees*)	*Frequency*
4— 6	6
6— 8	5
8—10	10
10—12	8
12—14	3
14—16	2
Total	34

Construct a cumulative frequency table for the data.

3. Construct a cumulative frequency table from the following and find the number of students scoring below 15 marks.

Marks	Frequency
3—7	3
7—11	10
11—15	12
15—19	8
19—23	5
23—27	9

[*C.B.S.E., 1984* (*A.I.*)]

(Section—B)

4. Construct a frequency table with equal class-intervals from the following data on the monthly wages (in rupees) of 28 labourers working in a factory, taking one of the class intervals as 210—230 (230 not included) :

220, 268, 258, 242, 210, 268, 272, 242, 311, 290, 300, 320, 319, 304, 302, 318, 306, 292, 254, 278, 210, 240, 280, 316, 306, 215, 258, 236.

5. The weekly observations on cost of living index in a certain city for a year give the following frequency table :

Cost of living index	*Number of weeks*
140—150	5
150—160	10
160—170	16
170—180	12
180—190	7
190—200	2
Total	52

Draw a histogram to represent the data.

6. The data below gives the monthly earnings of 38 workers in a flour mill.

Monthly earnings (in rupees)	Number of workers
500—520	4
520—540	6
540—560	12
560—580	8
580—600	6
600—620	2

Draw a frequency polygon of the above distribution. [*C.B.S.E., 1986* (*A.I.*)]

7. (*a*) Following is the distribution of marks of 70 students in a certain school test

Marks upto	Number of students
10	3
20	8
30	17
40	20
50	22

For the above data form a frequency distribution. Also find the number of students securing more than 40 marks.

(*b*) Represent the above data by a frequency polygon. [*C.B.S.E.*, *1982* (*A.I.*)]

(Section—C)

8. The monthly profits (in rupees) of 100 shops are distributed as follows :

Profit per shop	Number of shops
0— 50	12
50—100	18
100—150	27
150—200	20
200—250	17
250—300	6

Draw a histogram and the frequency polygon for the above data.

[*C.B.S.E.*, *1983* (*Delhi*)]

9. The following scores were obtained by 50 students in a test. Group the data into intervals of 5 each starting with the interval 37·5—42·5. From the grouped data make a frequency table, a histogram and cumulative frequency polygon.

88	74	67	49	69	38	86	77	66	75
94	67	78	69	84	50	39	58	79	70
90	79	97	75	98	77	64	69	82	71
65	68	84	73	58	78	75	89	91	62
72	62	74	81	79	81	86	78	90	81

[*C.B.S.E.*, *1984* (*Delhi*)]

10. Draw a cumulative frequency curve for the following data :

Marks	Frequency
0—10	15
10—20	10
20—30	22
30—40	40
40—50	15
50—60	8

11. The marks obtained (out of 70) by 50 students in an examination are given below :

Marks	Number of students
0—10	4
10—20	4
20—30	7
30—40	10
40—50	12
50—60	8
60—70	5

(*a*) **Draw a frequency polygon representing the above data.**

(*b*) **Draw also an ogive for the above data.**

□ □

11

BASIC GEOMETRICAL CONCEPTS

11.1. HISTORICAL BACKGROUND

The word *'Geometry'* is derived from the Greek words *'Geo'* meaning *'earth'* and *'metron'* meaning *'measuring'*. Ancient Egyptians were perhaps the first people to study Geometry. They were mainly concerned with problems of finding areas of rectilinear figures. Later Babylonians also studied the problems of finding areas of various rectilinear figures. Both the Egyptians and Babylonians used geometry mostly for practical purposes. However, they did not develop it as a systematic science.

The knowledge of Geometry passed from Egyptians to Greeks. The Greeks (600 B.C.—300 B.C.) collected and arranged the known facts in a system. Thales (600 B.C.—546 B.C.) was the first to treat geometry as a logical structure. The most famous pupil of Thales was **Pythagoras** (580 B.C.—500 B.C.). He gave to us many of the proofs of theorems given in our geometries of today.

Euclid who lived around 300 B.C. was another distinguished Greek mathematician, known as the father of Geometry. He started with a set of axioms, statements to be accepted as true, from which all theorems were deducted as logical sequences. Euclid's monumental work on Geometry is contained in his thirteen volume work called *'Elements'*.

Geometry of the Vedic period in India originated with constructions of various kinds of altars and fireplaces. For performing Vedic rites. **Aryabhata** (born A.D. 476) was a great Indian mathematician. He worked out the area of an isosceles triangle. Volume of a pyramid and gave a close approximate value of π. **Brahmagupta** (born A.D. 598) lived in Ujjain. He worked out the area of a cyclic quadrilateral in terms of its sides and semi-perimeter. **Bhaskara II** (born A.D. 1114) was another important Indian mathematician. He wrote the famous book known as *'Lilavati'* which is a treasure of mathematical knowledge. He gave a dissection proof of Pythagoras theorem. The excavations of Mohenjodaro, Harappa and Lothal testify to the extensive use of geometry by ancient Indians.

11.2. IMPORTANCE OF THE STUDY OF GEOMETRY

It is desirable to tell the importance of the study of geometry. Primarily, geometry teaches you how to reason. The habit of correct thinking acquired in its study is beneficial to all. By its study you will be able to converse more logically and read with a greater understanding.

After studying geometry, you will have a better appreciation of man's works and of the wonderful creations found in nature. Besides, you will have a greater interest in the arts and architecture which make use of geometric designs.

Later in your life, many of you will have to face situations where knowledge of geometry will be extremely useful and sometimes indispensable.

If you are thinking of becoming an engineer, an architect, a technician, a scientist or a mathematician, you must first learn geometry, since a knowledge of it is necessary for success in any of these professions. In fact, you cannot do in your profession without it.

11.3 NATURE OF FORMAL GEOMETRY

A logical structure of geometry consists of certain (1) undefined terms, (2) postulates, (3) definitions, and (4) theorems.

Self evident geometrical facts which are taken for granted. Without proofs, are called **postulates** or **axioms**.

A poperty is taken to be true when we arrive at it through logical reasoning based on axioms, definitions or undefined terms. This process is described as *proving* the properties. The results thus proved or the conclusions so obtained, are called **theorems.**

Plane Geometry is divided into two sections—

(*i*) **Theoretical** Geometry consisting of **Theorems,** and

(*ii*) **Practical** Geometry consisting of **Problems.**

Any separate geometrical operation, whether a discussion or a construction, is called a **proposition.**

Proposition is a common name for theorems and problems.

A theorem is a proposition in which a geometrical fact is proved by methodical reasoning.

A problem is a proposition in which a geometrical figure is to be actually constructed.

A proposition consists of three parts :

(*i*) the **enunciation,** (*ii*) the **construction,** (*iii*) the **proof**

(*i*) The enunciation can be divided into two parts :

(*a*) The general enunciation *i.e.*, the statement of a proposition in general terms.

(*b*) The particular enunciation *i.e.*, the same statement in special terms with reference to a diagram drawn for the purpose.

(*ii*) The construction is the drawing of lines, angles, circles, etc., which are necessary for the purpose of a theorem or a problem.

(*iii*) The proof is the process by which the required truth has been established or the required construction has been made.

The general enunciation of a theorem consists of two parts :

(1) the **hypothesis** or that which is given or assumed, and (2) the **conclusion** or that which is proved.

Likewise, the enunciation of a problem also consists of two parts :

(1) the **data,** meaning that which is given, and (2) the **quaesita,** meaning that which is to be constructed.

One theorem is said to be the **converse** of another when the hypothesis of one is the conclusion of the other and vice versa.

A **corollary** is a statement, the truth of which follows immediately from a theorem. It is, therefore, placed at the end of a theorem as an inference which requires no further proof.

All mathematical reasoning is based on **Definitions** and **Axioms**.

Definition is a precise statement explaining the exact meaning of word, term or phrase.

Axiom is a self-evident truth and is accepted without proof.

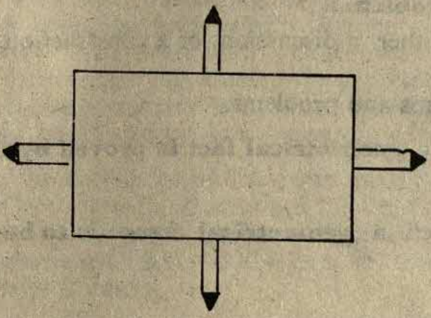
We will, now learn how to prove geometrical theorems.

11.4. POINT, LINE AND PLANE

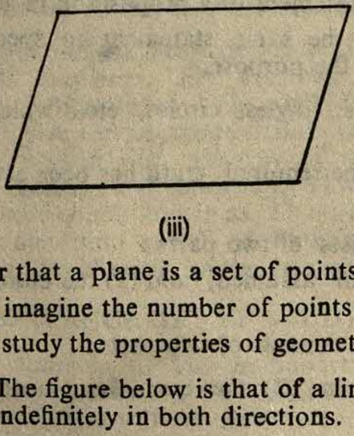
You are already familiar with the concepts of a point, line and plane. We accept these terms without definition and try to define other terms such as line-segment, ray, angle, etc., using these undefined terms. We shall now review these *undefined terms* and their relationship.

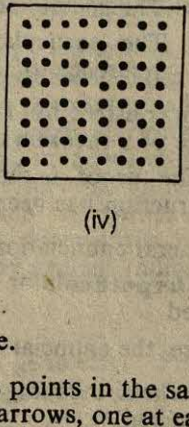
Point. A geometric point is an idea denoting position in space. We mark a dot (.) or a cross (×) to represent a point and use capital letters to name points.

Plane. To get an idea of a plane, think of the surface of a table, floor of the room, wall of the room, blackboard extended indefinitely in all directions. A plane has no thickness. A picture of a plane is usually made in the form (of a rectangle or parallelogram) as shown below. It should be understood that this represents only part of the plane.


(i)

(ii)


(iii)

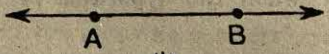

(iv)

It is clear that a plane is a set of points.

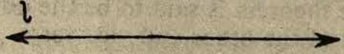
Can you imagine the number of points in a plane ?

We shall study the properties of geometric figures in a plane.

Line : The figure below is that of a line. A line represents points in the same direction. A line extends indefinitely in both directions. This is shown by arrows, one at each end. To name a line we select any two distinct points *C* and *D* any where on the line and read it as line *CD*. Sometimes we name a line by a single letter *l* or *m*. Symbolically we write $\overleftrightarrow{CD}$ or $\overleftrightarrow{BA}$.



(i)

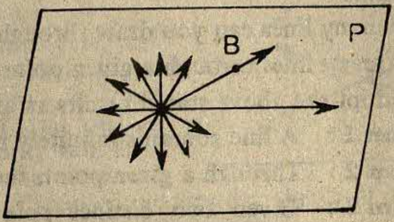


(ii)

Can you imagine the number of points on a line ?

Given a line l and a point P, if $P \in l$ *i.e.*, if P is an element of l, then we say that

P lies on l OR P is incident on l,
OR l contains P OR l passes through P.



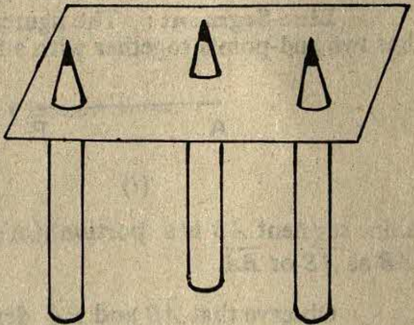
Take a point A in the plane P.

How many lines can you draw through A?

Mark another point B in the same plane.

How many lines can you draw through A and B?

We see that

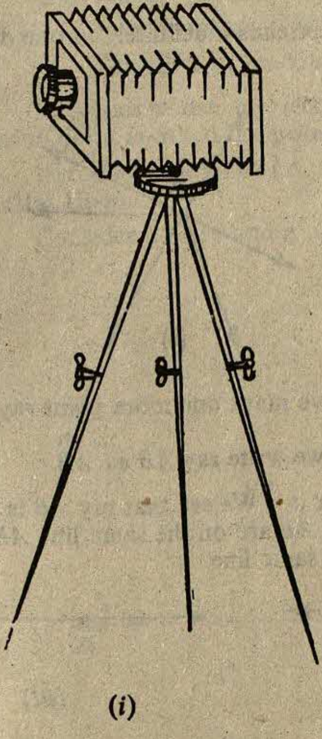


Two points determine one and only one line or exactly one line passes through two points.

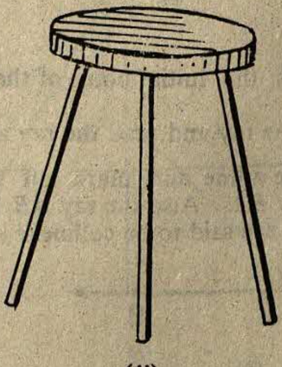
Points which lie on the same line are called collinear points, points that do not lie in one line are called non-collinear points.

Take a stiff piece of papper to represent a plane and use the points of pencils to represent points. Try to support the plane of paper on the points of pencils.

Can you do it with just (*a*) one pencil, (*b*) two pencils, (*c*) three pencils?



(*i*)



(*ii*)

Have you noticed that many scientific instruments (like camera, etc.) and items of furniture are mounted on tripods ?

We see that

Three non-collinear points determine a plane or every plane contains at least three non-collinear points.

How many lines can you draw through three non-collinear points ?

How many lines through a given point determine a plane ?

We adopt the above stated results as axioms—

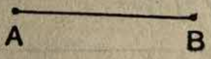
Axiom 1 : A line contains infinitely many points.

Axiom 2 : Through a given point, there pass infinitely many lines.

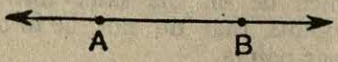
Axiom 3 : Given two distinct points A and B, there is one and only one line that contains both of the points.

11.5. LINE-SEGMENT AND RAY

Line-Segment : The figure (*i*) below represents a line-segment *AB*. Line segment *AB* has two end-points together with all the points lying between them. It can be easily seen that a line segment *AB* is a portion (or a subset) of the line *AB*. Symbolically we write segment *AB* as $\overline{AB}$ or $\overline{BA}$.



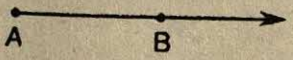
(*i*)



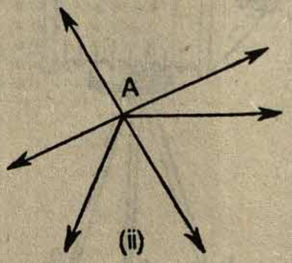
(*ii*)

Observe that $\overline{AB}$ and $\overline{BA}$ denote the *same* line segment, since both are the same set of points. Also, a line segment has a definite length whereas a line does not have a definite length.

Ray : The figure below represents a ray *AB*. A ray extends indefinitely in one direction We show this by an arrow.



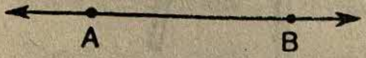
(i)



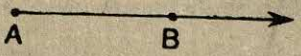
(ii)

A is the initial point of the ray. To name a ray we mark one more point say *B* any. where on the ray and read the ray as ray *AB*. Symbolically we write ray *AB* as $\overrightarrow{AB}$.

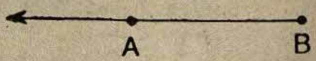
Take a line and mark off two points *A* and *B* on it. We see that ray *AB* is not the same as ray *BA*. Also the ray *AB*, the ray *BA*, the segment *AB* are on the same line *AB*. Rays or segment are said to be collinear if they are subsets of the same line.



(*i*)

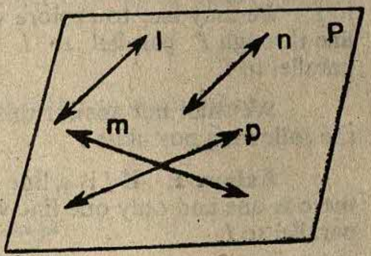


(*ii*)



(*iii*)

11.6. PARALLEL AND INTERSECTING LINES



Imagine lines drawn in a plane. Lines *m* and *p* have a point in common *i.e.*, they intersect in one point. Lines *l* and *p* (which do not intersect in one point) are parallel.

What about lines (*i*) *l* and *m* (*ii*) *m* and *n* ?

Two lines which lie in the same plane (*i.e.*, are coplanar) and do not intersect are said to be parallel to each other. Note the phrase 'in the same plane'. In space two lines may also be *skew*. Lines are skew if and only if they are non-coplanar. The line where the north wall of a room intersects the ceiling and the line where the west wall intersects the floor could represent skew lines. Extended indefinitely they will never intersect in a single point. They are not parallel because they are not coplanar.

Parallel lines are very useful in our daily life. Some common examples of parallel lines are : the railway lines, opposite edges of a table or book, telephone wires between two poles, etc.

THEOREM—1

Two distinct lines cannot have more than one point in common.

Given : Two distinct lines *l* and *m*.

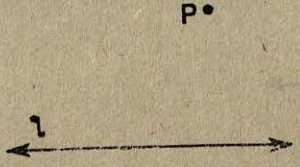
To prove : $l \cap m$ contains at most one point.

Proof : Suppose $l \cap m$ contains two distinct points *A* and *B*.
Then line *l* contains the points *A* and *B*.
Line *m* also contains the points *A* and *B*.
Hence, it follows from axiom 3 that $l = m$.
This is a contradiction, since we have supposed that *l* and *m* are two distinct lines.
Hence, our supposition is wrong.
Therefore, $l \cap m$ contains at the most one point.

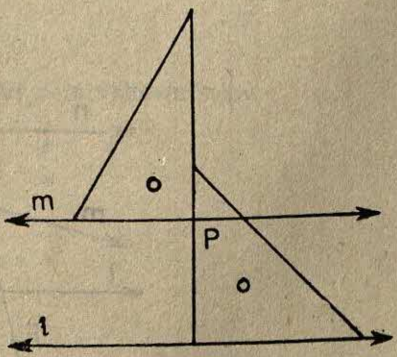
Two lines whose intersection is non-empty are said to be **intersecting lines**. The common point is called the **point of intersection.**

Two lines in a plane *i.e.*, two coplanar lines whose intersection is empty are said to be **parallel lines.**

Consider a line *l* and a point *P* not on the line *l*.

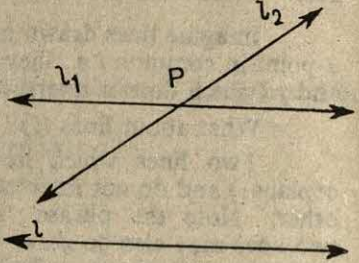


(i)



(ii)

With the help of set squares draw a line l_1 (through P) parallel to line l. Thus we have one line through P parallel to l We may like to explore whether there is more than one line through P parallel to l. Perhaps l_1 and l_2 are both parallel to l.

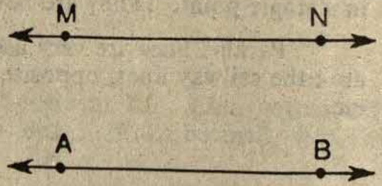


We shall not answer this question but we shall accept the following postulate :

Axiom 4. If l is a line and P is a point not on l, there is one and only one line which passes through P and is parallel to l.

11.7. SOME DEFINITIONS

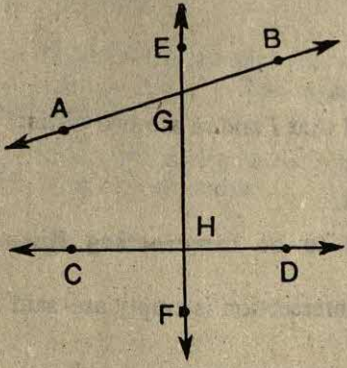
1. **Parallel segments**. Segments are parallel if and only if the lines containing them are parallel. Segment MN is parallel to segment AB because line MN is parallel to line AB.



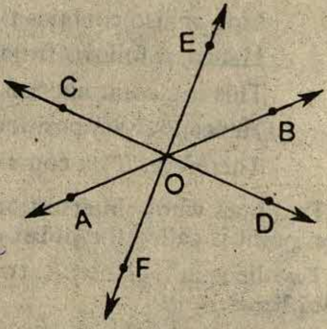
Symbolically we write $\overline{MN} \parallel \overline{AB}$. or segment $MN \parallel$ segment AB.

2. **Parallel rays.** Two rays are said to be parallel if the lines containing them are parallel.

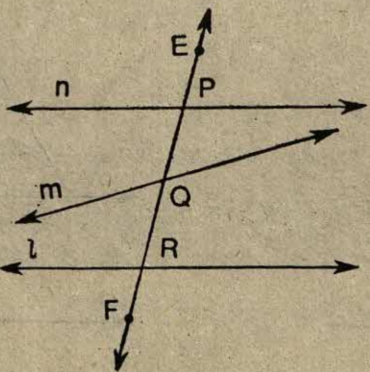
3. **Transversal.** A line that intersects two or more (coplanar) lines in distinct points is called a transversal. In Fig. (*i*) EF is a transversal as it intersects two lines AB and



(i)



(ii)

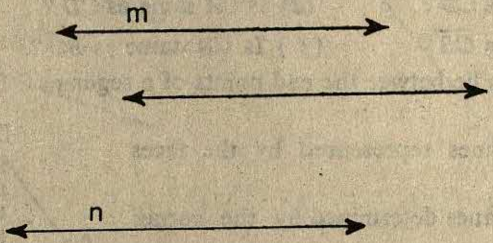


(iii)

CD in two distinct points *G* and *H*. In Fig. (*ii*) *EF* is not a transversal as it intersects two lines *AB* and *CD* in only one point *O*. In Fig. (*iii*) transversal *EF* intersects three lines in three distinct points *P*, *Q* and *R*.

THEOREM—2

Two lines which are both parallel to the same line, are parallel to each other.



Given : Three lines l, m, n in a plane such that $l \parallel m$ and $n \parallel m$

To prove : $l \parallel n$.

Proof : Suppose l is not parallel to n.

Then lines l and n intersect in a unique point, say P.

Now P is not on line m, because P is on line l and $l \parallel m$.

$\therefore$ Through a point P outside m, there are two lines l and n, both parallel to line m.

By parallel axiom, this is impossible.

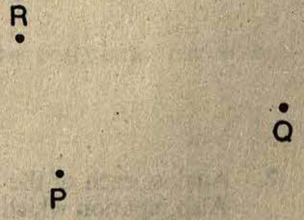
$\therefore$ Our supposition is wrong.

Hence, line l is parallel to line n.

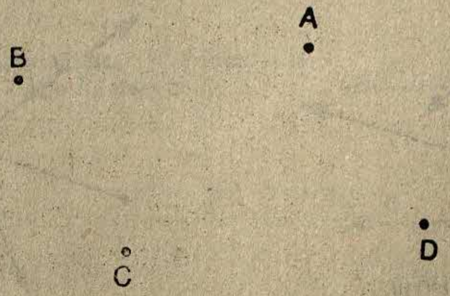
EXERCISE—11(a)

(Section—A)

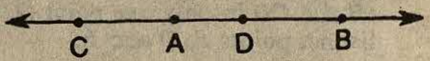
1. Name all the lines you can draw through three points, shown on the right.



2. Name all the lines you can draw through four points shown below :



3. Look at the figure drawn on the right and answer the following questions.



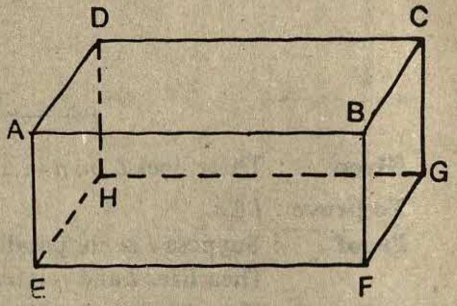
(*a*) Is $\overrightarrow{CB}$ same as $\overrightarrow{CD}$? (*b*) Is $\overrightarrow{AB}$ same as $\overrightarrow{BA}$?

(*c*) Is $\overleftrightarrow{AD}$ same as $\overleftrightarrow{CB}$? (*d*) Is $\overrightarrow{CA}$ same as $\overrightarrow{CD}$?

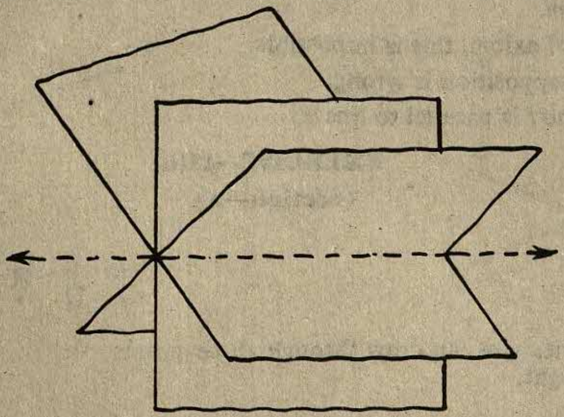
(*e*) Is $\overline{AD}$ same as $\overline{CB}$? (*f*) Is $\overline{CB}$ same as $\overline{BC}$?

4. How many points lie betwee the end points of a segment ?

5. (*a*) Name the planes represented by the faces of the box.
   (*b*) Name the planes determined by the corner points.
   (*c*) Name two lines that do not interset but lie in same plane.
   (*d*) Name two lines that do not intersect and do not lie in same plane.



6. How many planes can pass through one line ? See the figure and answer.



7. Against each of the following figures (*i*) to (*vi*), write intersecting or non-intersecting. Also mention which of the intersecting ones are concurrent and non-intersecting are parallel.

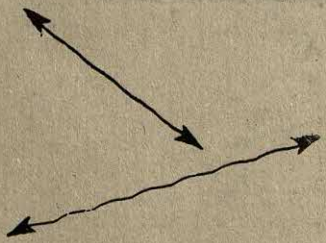


Fig (i)

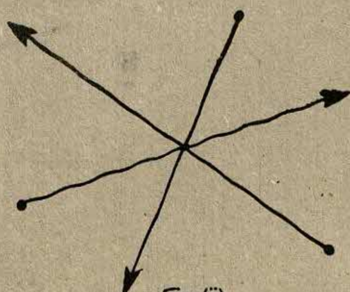


Fig (ii)

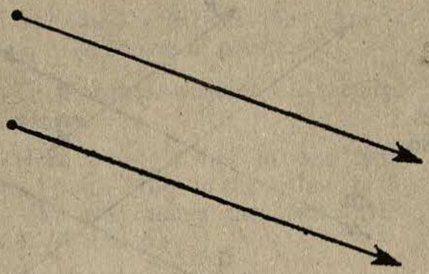


Fig (iii)

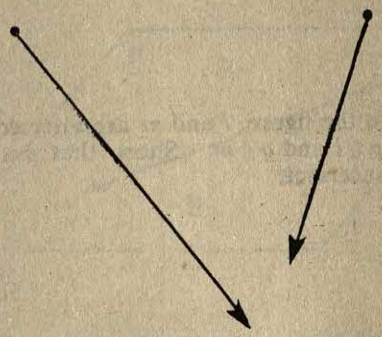


Fig (iv)

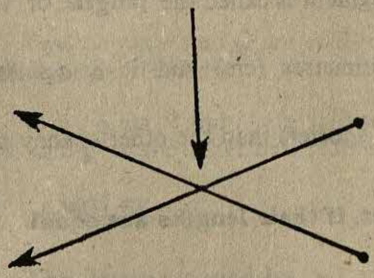


Fig (v)

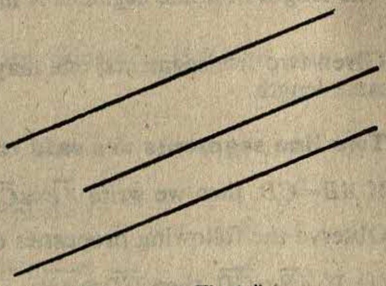


Fig (vi)

Section—B

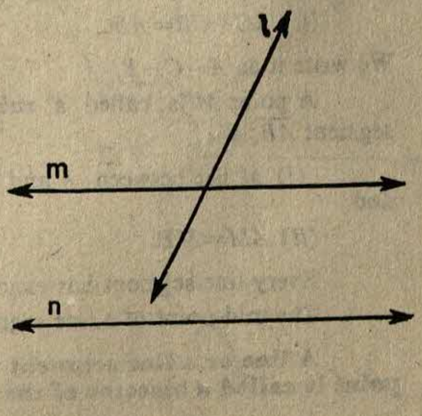
8. Given a line, do you think the line is a subset of at least one plane ? Is there only one plane of which it is a subset ?

9. Given three points not in a line, do you think there exists at least one plane containing all the three points ?

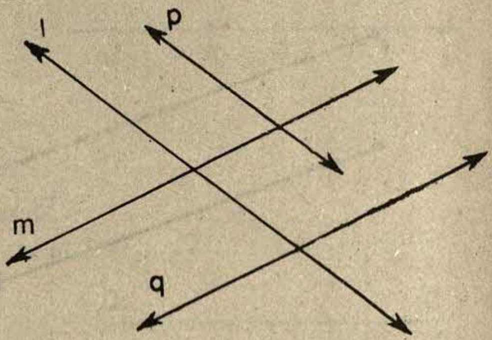
   Is there more than one such plane ?

Section—C

10. *l*, *m*, *n* are lines in the same plane. If *l* intersects *m* and *n* ∥ *m*, show that *l* also intersects *n*.



11. In the figure, l and m are intersecting lines. $p \parallel l$ and $q \parallel m$. Show that p and q also intersect.



11 8. CONGRUENCE OF LINE SEGMENTS

The distance between the end points of a line segment is called the **length** of the line segment.

The length of a line segment is measured in centimetres (cm) and it is *a positive real number.*

Given two line segments, one may be longer (or shorter) than the other or they may be of the same length.

Two line segments are said to be congruent, if their lengths are equal.

If $AB=CD$, then we write $\overline{AB} \cong \overline{CD}$.

Observe the following properties of *congruence of line segments*—

(1) If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$.

(2) Every line segment is congruent to itself *i.e.*, $\overline{AB} \cong \overline{AB}$.

(3) If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$.

Remember that two congruent line segments have equal lengths, and conversely two line segments of equal length are congruent.

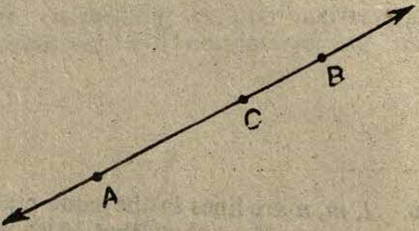
11.9. BETWEENNESS

A point C is said to lie **between** A and B, if

(*i*) A, B and C are collinear, and

(*ii*) $AC+CB=AB$

We write it as $A-C-B$.



A point M is called a **mid-point** of a line segment $\overline{AB}$, if

(*i*) M lies between A and B *i.e.*, $A-M-B$,

and

(*ii*) $AM=MB$.

Every line segment has exactly one mid-point.

The mid-point of a line segment is said to **bisect** the line segment.

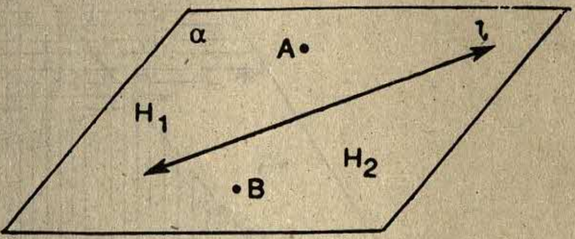
A line or a line segment which intersects a given line segment in its mid-point is called a bisector of the line-segment.

11.10. HALF PLANE

The line l separates the plane α into two disjoint sets of points.

Each of these disjoint sets is called a **half-plane.**

Here point A lies in one half-plane and point B lies in the other half-plane. We may name the two half-planes as H_1 and H_2.



The line l is called the **edge** of each half-plane.

Given a line l, in a plane α, the points of α. other than those on l, form two disjoint subsets of the plane. Each of these parts is called a half-plane, with edge in l.

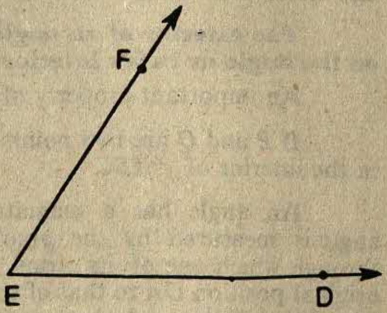
Two points P and Q are said to be **on the same side** of l, if P and Q are in the *same* half-plane separated by l.

Two points P and Q are said to be **on opposite sides** of l, if P is in one of the half-planes, and Q in the other half plane, separated by l.

11.11. ANGLES

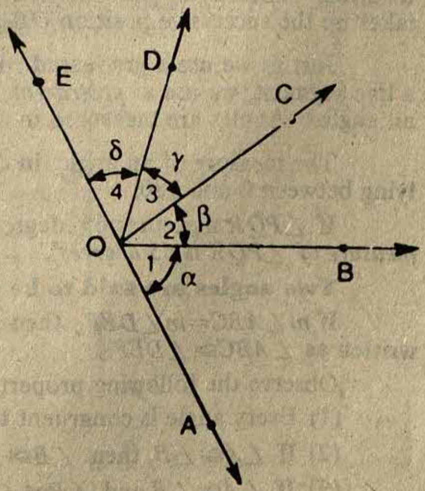
The figure formed by two rays with a common initial point is called an angle. The two rays are called the **arms** of the angle and the common initial point is called the **vertex** of the angle.

An angle is usually denoted by three letters naming the vertex and one point on each of the arms. The letter naming the vertex is written in the middle. Thus, figure on the right denotes the angle FED or angle DEF. We use the symbol '$\angle$' for an angle and write $\angle FED$ or $\angle DEF$.



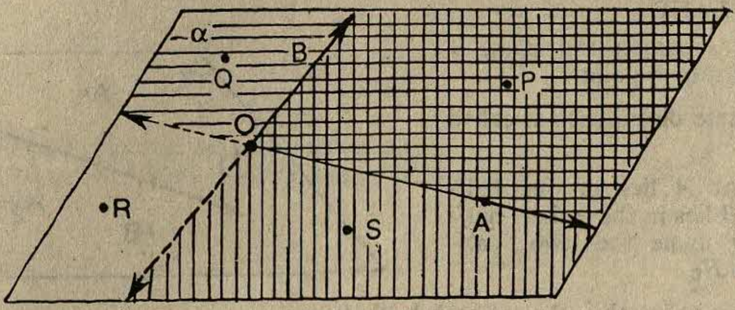
Angle between two segments would mean angle between the rays containing the segments.

If there is only one angle at the vertex, it is convenient to denote it by a single letter at the vertex. Thus $\angle DEF$ can be named as $\angle E$. Sometimes a digit or a letter from Greek alphabet is written between the arms to denote an angle. For example, in the figure shown on the right the angles are 1, 2, 3, 4. The Greek letters generally used are α (Alpha), β (Beta), γ (Gama), δ (Delta) θ (Theta), etc.,



In the figure drawn below, $\angle AOB$ is in the plane α. The rays OA and OB determine $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.

Line OA separates the plane α into two half-planes, one of which contains point B. Let us shade that half-plane with *horizontal* lines.



Line OB separates the plane α into two half-planes, one of which contains point A. Let us shade that plane with *vertical* lines.

The *intersection* of these two half-planes is called the **interior of $\angle AOB$**. The crossed portion shows the interior of $\angle AOB$.

Here point P is in the *interior* of $\angle AOB$ while points Q, R and S are in the *exterior* of $\angle AOB$.

Observe that any point of the plane α is either in the interior of $\angle AOB$ or on $\angle AOB$ or in the exterior of $\angle AOB$. Points of $\angle AOB$ *do not* belong either to the interior or to the exterior.

The interior of an angle BAC is the set of all points in its plane which lie on the same side of the line AB, as C ; and also on the same side of line AC as B.

The exterior of an angle BAC is the set of points in its plane which do not lie on the angle or in its interior.

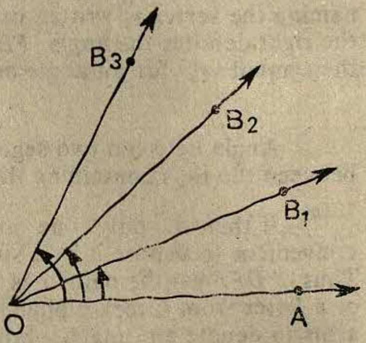
An important property of the interior of an angle is as under :

If P and Q are two points in the interior of $\angle ABC$, then line segment PQ is contained in the interior of $\angle ABC$.

An angle has a magnitude. The magnitude of an angle is measured by the amount of turning or rotation through which one of its arms, say OA, revolves from its original position OA to that of the other arm, say OB_1. The magnitude of the angle does not depend on the lengths of its arms. The angle goes on increasing as the moving arm takes up the successive position OB_2, OB_3.

Just as we use a graduated ruler to find the length of a line segment, we use a *protractor* to find the measure of an angle. Angles are measured in *degrees*.

The measure of an angle, in degrees, is a *real number* lying between 0 and 180.



If $\angle PQR$ measures 65 degrees, we write it as $m\angle PQR = 65$. We read it as '*degree measure of $\angle PQR$ is 65 degrees*'.

Two angles are said to be congruent if they have the same measure.

If $m\angle ABC = m\angle DEF$, then $\angle ABC$ and $\angle DEF$ are said to be congruent. This is written as $\angle ABC \cong \angle DEF$.

Observe the following properties of congruence of angles :

(1) Every angle is congruent to itself *i.e.*, $\angle ABC \cong \angle ABC$.

(2) If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.

(3) If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

11.2. KINDS OF ANGLES

Look at the following angles and think about the measure each of them.

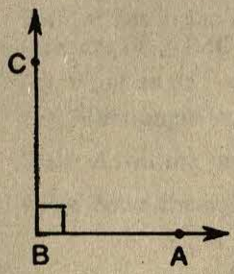


Fig (i)

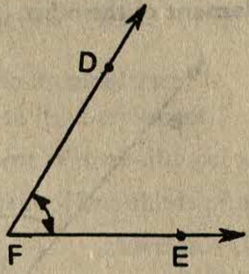


Fig (ii)

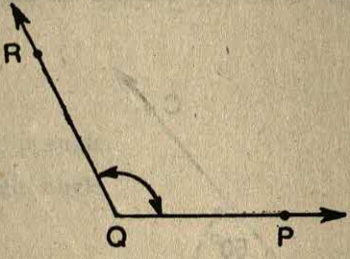


Fig (iii)

1. **An angle whose measure is 90° is called a right angle.** See Fig. (*i*).
2. **An angle whose measure is less than 90° is called an acute angle.** See Fig. (*ii*).
3. **An angle whose measure is more than 90° is called an obtuse angle.** See Fig. (*iii*).

If the sum of the measures of two angles is equal to 90, the angles are said to be complementary angles.

Then each angle is called the **complement** of the other.

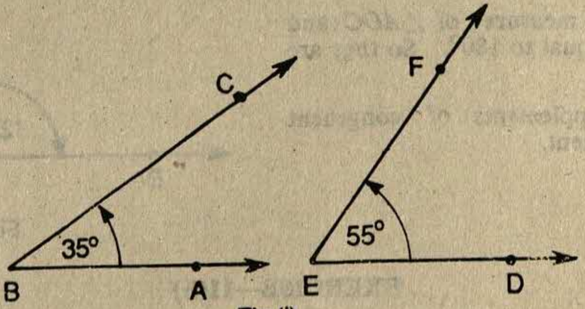


Fig (i)

In Fig. (*i*), the measures of $\angle ABC$ and $\angle DEF$ are together equal to 90°. So they are complementary angles and each of them is the complement of the other.

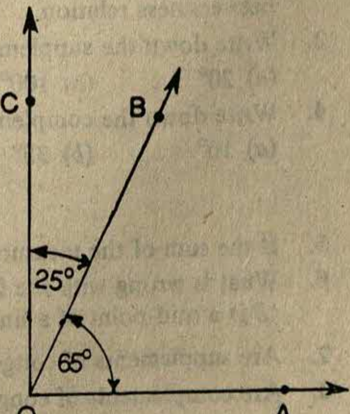


Fig (i)

In Fig. (*ii*), the measures of $\angle AOB$ and $\angle BOC$ are together equal to 90°. So they are complementary angles.

Note that complements of congruent angles are also congruent.

If the sum of the measures of two angles is equal to two right angles, the angles are said to be supplementary angles.

Then each angle is called the **supplement** of the other.

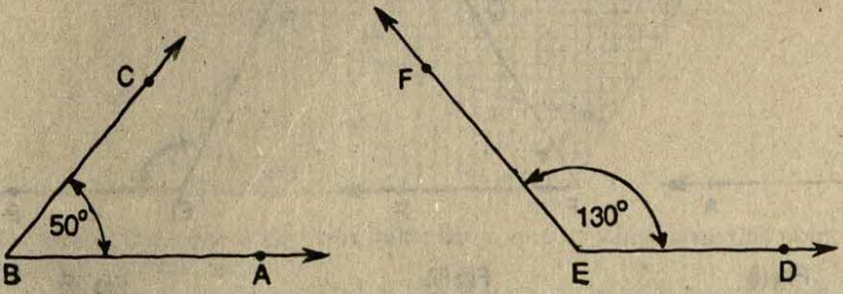


Fig (i)

In Fig. (*i*), the measures of $\angle ABC$ and $\angle DEF$ are together equal to 180°. So they are supplementary angles and each of them is the supplement of the other.

In Fig. (*ii*), the measures of $\angle AOC$ and $\angle BOC$ are together equal to 180°. So they are supplementary angles.

Note that supplements of congruent angles are also congruent.

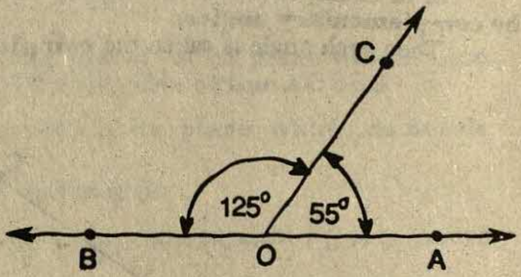


Fig (ii)

EXERCISE—11(b)

(Section—A)

1. If B lies between A and C and $AC=10$ cm. $BC=6$ cm. find AB.
2. Given A, B, C are collinear points : $AB=7$ cm, $BC=3$ cm and $AC=4$ cm. Write the betweenness relation.
3. Write down the supplements of the following angles :
   (*a*) 20° (*b*) 100° (*c*) 72° (*d*) 156°
4. Write down the complements of the following angles :
   (*a*) 10° (*b*) 35° (*c*) 77° (*d*) 53°

(Section—B)

5. If the sum of the measures of three angles is 180°, are the angles supplementary ?
6. What is wrong with the following description of the mid-point of a line segment ?
   'B is a mid-point of a line segment, $\overline{AC}$, if $AB=BC$'.
7. Are supplements of congruent angles always congruent ?
8. Are complements of congruent angles always congruent ?

9. Are the interior and exterior of a given angle always disjoint sets ?

10. Point P lies in the interior of $\angle BAC$. If $\angle BAC = 72°$ and $\angle BAP = 47°$, what is the measure of $\angle PAC$?

11. (*a*) What angle is equal to its supplement ?

    (*b*) What angle is equal to its complement ?

12. Write down the supplement of three-fifths of a right angle.

13. Write down the complement of two-thirds of a right angle.

(Section—C)

14. How many degrees are there in an angle which equals one-fifth of its supplement ?

15. One of two complementary angles is seven-eighths as large as the other. How many degrees are in each angle ?

16. If C is a point on line $\overleftrightarrow{AB}$ such that $AC = CB$, then prove that C lies between A and B.

12

ANGLE RELATIONS

12.1. Many times we come across two or more angles having the same vertex. There angles possess certain relations among themselves. Examine the following figures :

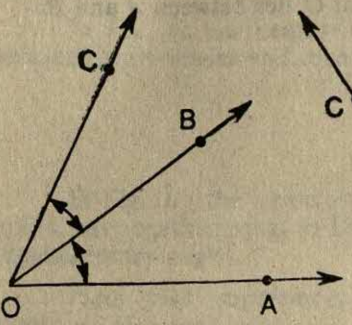


Fig (i)

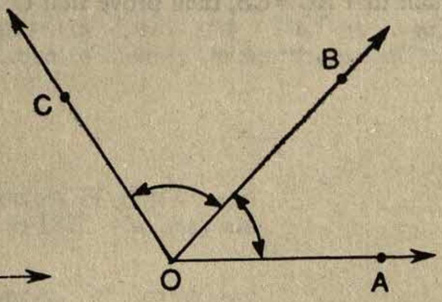


Fig (ii)

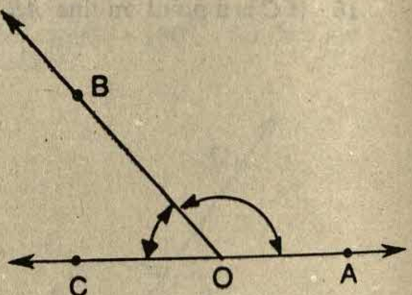


Fig (iii)

Here three rays *OA*, *OB* and *OC* have a *common initial point O.* These rays form two angles, ∠*AOB* and ∠*BOC* with a *common vertex O.* They have a *common arm*, ray *OB*. The distinct arms, $\overrightarrow{OA}$ and $\overrightarrow{OC}$ lie on *different sides* of ray *OB*. Such angles are called **adjacent angles.**

Thus, ∠*AOB* and ∠*BOC* are adjacent angles.

Examine the interiors of ∠*AOB* and ∠*BOC*.

Do they overlap each other ?

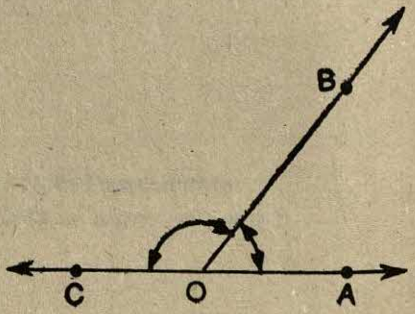
Are they disjoint sets ?

Two angles are called adjacent angles, if

(*i*) **they have the same vertex.**

(*ii*) **they have a common arm, and**

(*iii*) **their other arms lie on the opposite sides of the common arm.**



Look at the adjacent angles drawn on the right.

Measure ∠*AOB* and ∠*BOC.*

What is the sum ofthe measures of these angles ?

Are these supplementary angles ?

$$\angle AOB + \angle BOC = 180°$$

Thus, $\angle AOB$ and $\angle BOC$ are *adjacent supplementary angles.*

Such angles are said to form a **linear pair.**

Examine their *non-common arms,* rays OA and OC.

We see that non-common arms, $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are *opposite rays.* Also points A, O and C are collinear.

Two adjacent angles are said to form a linear pair of angles, if their non-common arms are two opposite rays.

Look at the figures drawn below :

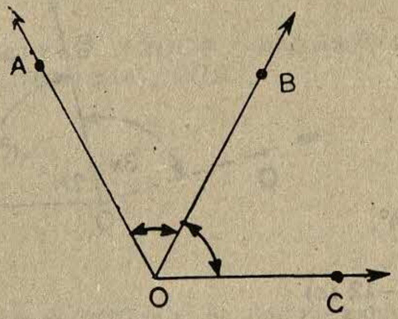


Fig (i)

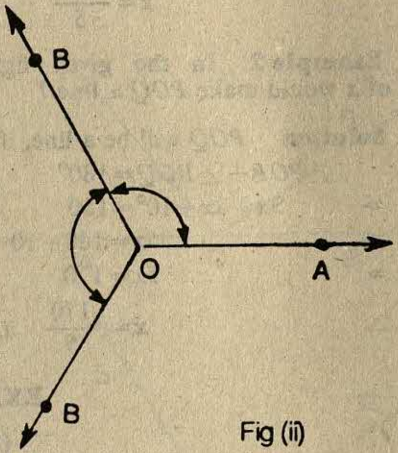


Fig (ii)

Measure the angles in both the figures. We see that

In Fig. (*i*), $\angle AOB + \angle BOC < 180°$.

In Fig. (*ii*), $\angle AOB + \angle BOC > 180°$.

Therefore, $\angle AOB$ and $\angle BOC$ *do not* form a linear pair in both the cases.

Hence, we have arrived at an important relation between two adjacent angles.

If $\angle AOB$ and $\angle BOC$ form a linear pair, then

$$\angle AOB + \angle BOC = 180°,$$

and if $\angle AOB$ and $\angle BOC$ do not form a linear pair, then

$$\angle AOB + \angle BOC \neq 180°.$$

We adopt this geometrical fact as an axiom.

AXIOM

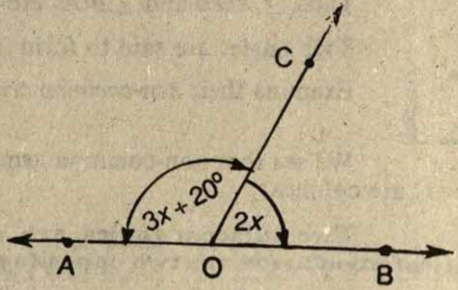
If a ray stands on a line, then the sum of the two adjacent angles so formed is 180°.

Conversely, if the sum of two adjacent angles is 180°, then the non-common arms of the angles are two opposite rays.

We have already learnt about supplementary angles. So we can restate linear pair axiom with the help of the definition of supplementary angles.

Two adjacent angles form a linear pair, if and only if they are supplementary.

Example 1. In the adjoining figure, if AOB is a line, find x.

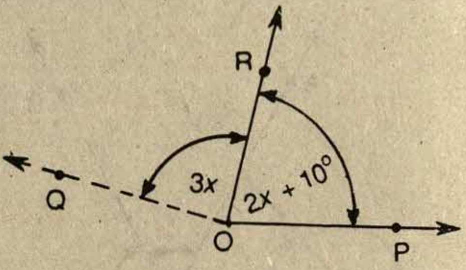


Solution. Here AOB is a line and ray OC stands on it.

So, by linear pair axiom, we have

$$\angle AOC + \angle COB = 180°$$
$$\Rightarrow \quad 3x+20+2x=180$$
$$\Rightarrow \quad 5x=180-20$$
$$\Rightarrow \quad 5x=160$$
$$\therefore \quad x=\frac{160}{5} \quad i.e., \ x=32°.$$

Example 2. In the given figure, what value of x would make POQ a line ?



Solution POQ will be a line, if

$$\angle POR + \angle ROQ = 180°$$
$$\Rightarrow \quad 3x+2x+10°=180$$
$$\Rightarrow \quad 5x=180-10$$
$$\Rightarrow \quad 5x=170$$
$$\therefore \quad x=\frac{170}{5} \quad i.e., \ x=34°.$$

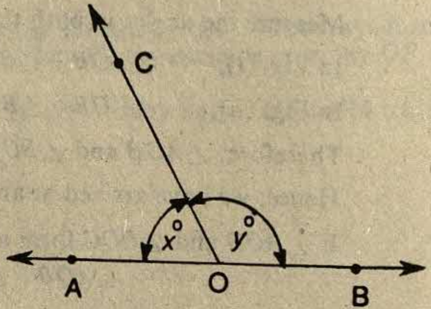
EXERCISE—12 (*a*)

(Section—A)

1. In the adjoining figure, OA and OB are opposite rays.

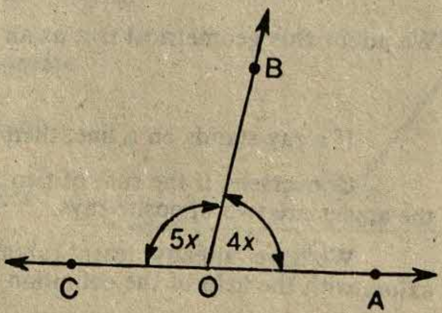
   (*a*) If $x=62°$, what is the value of y ?

   (*b*) If $y=105°$, what is the value of x ?



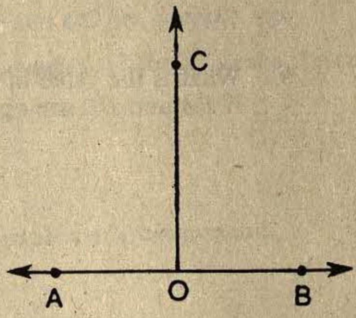
2. Two lines AB and CD meet at C. Find the magnitude of $\angle ACD$, if $\angle DCB$ is equal to
   (*a*) 77° (*b*) $m°$.

3. In the adjoining figure, $\angle AOB$ and $\angle BOC$ form a linear pair. Determine the value of x.

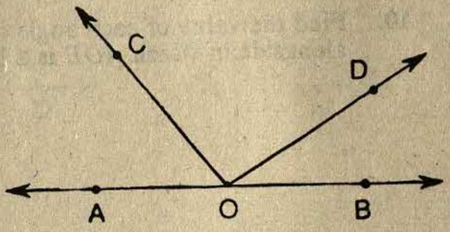


(Section— B)

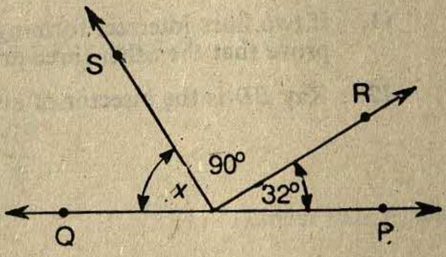
4. If ray OC stands on line AB such that
$$\angle AOC = \angle BOC,$$
show that each angle is a right angle.



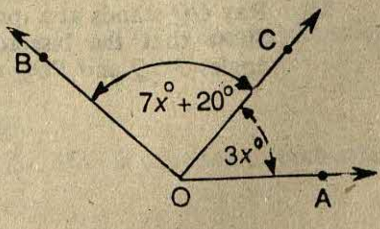
5. In the figure, if $\angle AOC + \angle BOD = 80°$, then find the measure of $\angle COD$.



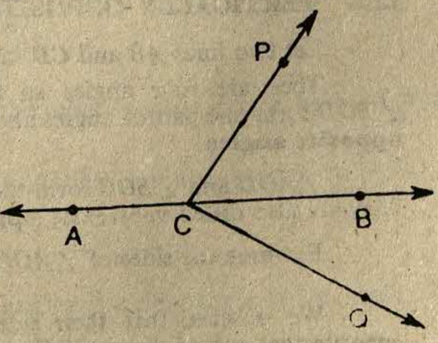
6. In the figure, POQ is a line, find the value of x.



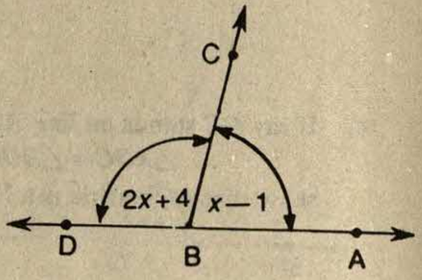
7. In the figure along side, if
$$\angle BOC = 7x° + 20° \quad \text{and} \quad \angle COA = 3x°,$$
find the value of x for which AOB becomes a line.



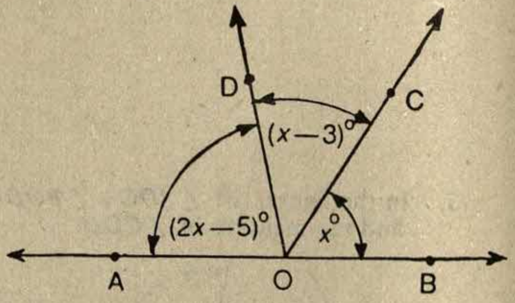
8. In the figure, AB is a line and rays CP and CQ are drawn such that the ray CB bisects $\angle PCQ$. Show that $\angle PCA = \angle QCA$.



9. What is the value of 'x' in the figure alongside, if BA and BD are opposite rays ?



10. Find the value of each angle in the figure alongside in which AOB is a line.



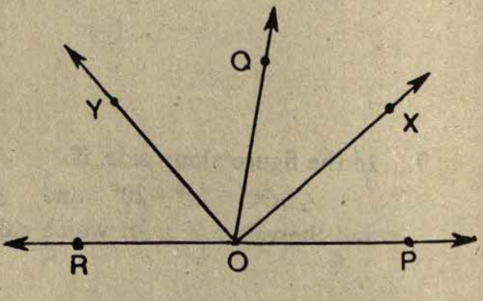
(Section—C)

11. If two lines intersect forming four angles, of which one is known to be a right angle, prove that the other three are also right angles.

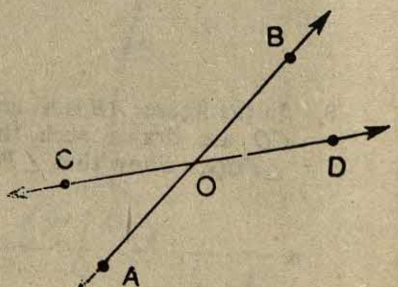
12. Ray BD is the bisector of any $\angle ABC$ and DB is produced to E ; prove that

$$\angle ABE = \angle CBE.$$

13. Ray OQ stands at a point O on line RP. Show that the bisectors of the adjacent angles POQ and ROQ contain a right angle.



12.2. VERTICALLY OPPOSITE ANGLES



Let two lines AB and CD intersect at O.

There are four angles so formed. $\angle AOC$ and $\angle BOD$ form one pair of angles and are called **vertically opposite angles**.

$\angle AOD$ and $\angle BOC$ form the other pair of angles. They are also called **vertically opposite angles**.

Examine the sides of $\angle AOC$ and $\angle BOD$.

We observe that their sides, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ form *opposite rays* and so do rays OC and OD.

Again examine the sides of $\angle AOD$ and $\angle BOC$.

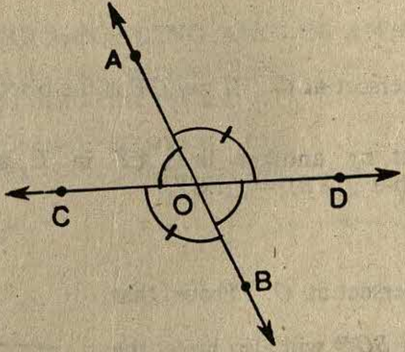
We observe that $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are *opposite rays* and $\overrightarrow{OD}$ and $\overrightarrow{OC}$ are *opposite rays.*

Two angles are called a pair of vertically opposite angles, if their arms form two pairs of opposite rays.

How are the two angles of a pair related ?

THEOREM—3

If two lines intersect, then the vertically opposite angles are congruent.



Given : Two lines AOB and COD intersecting at a point O.

To prove : (*i*) $\angle AOC=$vert. opp. $\angle DOB$
and (*ii*) $\angle AOD=$vert. opp. $\angle COB$

Proof : Ray OA stands on the line CD at O.

$\therefore \quad \angle AOC+\angle AOD=180^\circ$ ...(1) **(Linear Pair Axiom)**

Again ray OD stands on the line AB at O.

$\therefore \quad \angle AOD+\angle DOB=180^\circ$ ...(2) **(Linear Pair Axiom)**

From (1) and (2), we get

$$\angle AOC+\angle AOD=\angle AOD+\angle DOB.$$

$\therefore \quad \angle AOC=\angle DOB$ **($\angle AOD$ is common to both sides).**

Similarly, $\angle AOD=\angle COB$ proved.

EXERCISE—12 (*b*)

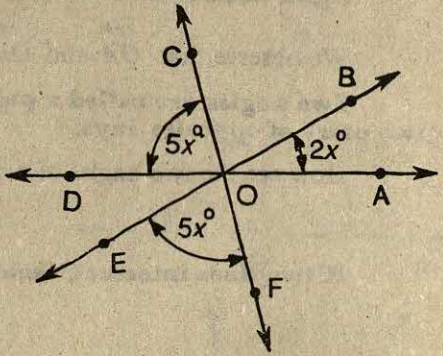
(Section—A)

1. If two lines intersect at an angle of 40°, find all the other angles at the point of intersection.
2. In the figure of the above theorem, if $\angle AOD=2\ \angle DOB$, find the magnitude of all the angles.

(Section—B)

3. If two straight lines intersect so that one of the angles is a right angle, prove that all the other angles are also right angles.

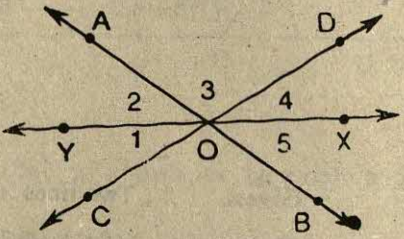
4. In the adjoining figure, determine the value of x.



5. Two lines AB and CD intersect at O. If ray OE is the bisector of $\angle AOC$, prove that ray EO bisects $\angle BOD$.

6. The lines AD, AE are cut by another line XY in B and C respectively so that $\angle ABC = \angle ACB$; show that $\angle XBD = \angle ECY$.

7. Two lines AB and CD intersect at O. Prove that the bisector $\overrightarrow{OX}$ of the $\angle BOD$ will also bisect the $\angle AOC$, if produced backwards.

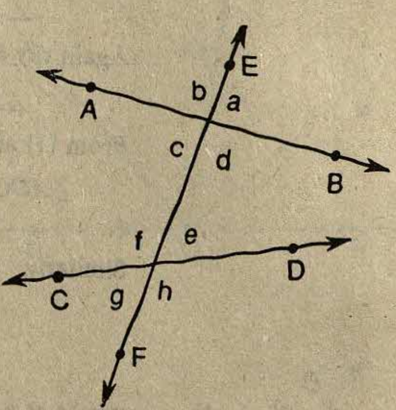


(Section—C)

8. Prove that the bisectors of vertically opposite angles are in the same line.

12.3. ANGLES MADE BY A TRANSVERSAL

When a transversal, $\overleftrightarrow{EF}$ cuts two lines AB and CD, eight angles are formed. Four of the angles are formed at one point of intersection and the remaining four angles are formed at the other point of intersection.



By virtue of the positions they occupy in the figure, some of them can be paired together as under :

(*i*) Angles a, b, g, h are called **exterior angles**. They are formed on the outer sides of lines AB and CD.

(*ii*) Angles c, d, e, f are called **interior angles**. They are formed on the inner sides of lines AB and CD *i.e.*, between lines AB and CD.

(*iii*) Angles c and e, d and f are pairs of **alternate interior angles**. They are formed on the opposite sides of the transversal and between lines AB and CD.

(*iv*) Angles a and e, b and f, c and g, d and h are called pairs of **corresponding angles**. They are formed on the same side of transversal, one of them being interior and the other exterior.

In general, there may be no relation between the angles of any of these pairs. However, when the lines are *parallel*, there exists some useful relations between the angles of each pair.

We are already familiar with the following facts, obtained by practical experience in the previous classes.

When a transversal intersects two parallel lines, then

(1) each pair of corresponding angles are equal,
(2) each pair of alternate interior angles are equal, and
(3) each pair of interior angles on the same side of the transversal are supplementary.

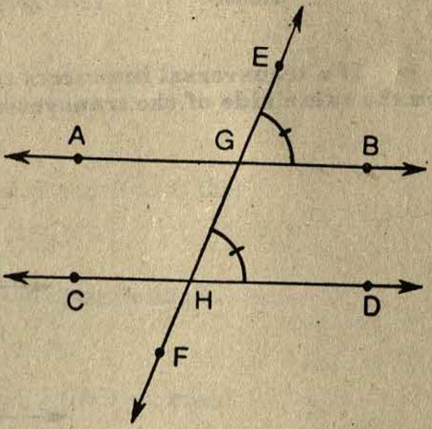
Can we prove these geometrical facts ?

So far we have learnt theorems about intersecting lines. We cannot prove these facts about parallel lines with their help. So, we agree to have the first result stated above as the *axiom*.

CORRESPONDING ANGLES AXIOM

If a transversal intersects two parallel lines, then each pair of corresponding angles are equal.

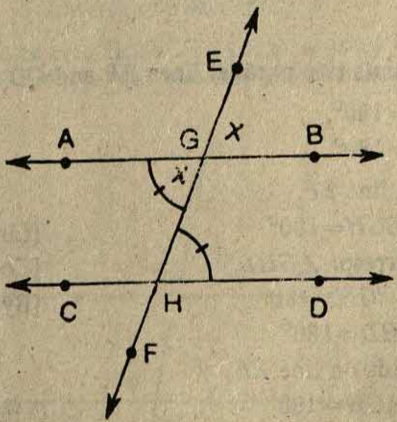
Conversely, if a transversal intersects two lines, making a pair of corresponding angles equal, then the lines are parallel.



Now we can deduce other facts about parallel lines and their transversal.

THEOREM—4

If a transversal intersects two parallel lines, then each pair of alternate interior angles are equal. [*C.B.S.E., 1983 (A.I.)*



Given : A transversal, $\overleftrightarrow{EF}$ cuts two parallel lines AB and CD at G and H respectively.

To prove : $\angle AGH$=alternate $\angle GHD$

$\angle BGH$=alternate $\angle GHC$

Proof : $\angle AGH = \angle EGB$ [Vertically opposite angles

$\angle LGB =$ corresp. $\angle GHD$ [Corresponding Angles Axiom

$\therefore \angle AGH = \angle GHD$

Ray GH stands on line AB.

$\therefore \angle AGH + \angle BGH = 180°$ [Linear Pair Axiom

Then $\angle BGH = 180° - \angle AGH$ ...(1)

Ray HG stands on line CD.

$\therefore \angle GHC + \angle GHD = 180°$

Then $\angle GHC = 180° - \angle GHD$ ...(2)

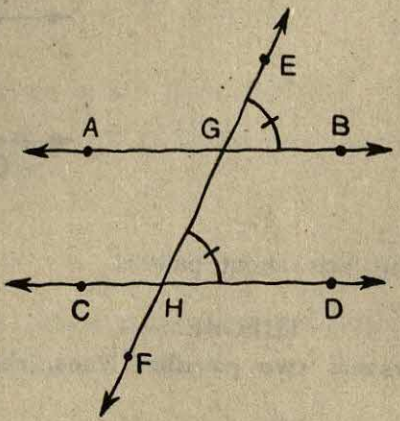
From (1) and (2), we have

$\angle BGH = \angle GHC$ [$\angle AGH = \angle GHD$ proved.

Hence $\angle AGH = \angle GHD$ and $\angle BGH = \angle GHC$.

THEOREM—5

If a transversal intersects two parallel lines, then each pair of interior angles on the same side of the transversal are supplementary.



Given : A transversal, $\overleftrightarrow{EF}$ cuts two parallel lines AB and CD at G and H respectively.

To prove : $\angle BGH + \angle GHD = 180°$,

$\angle AGH + \angle GHC = 180°$.

Proof : Ray GB stands on line EF.

$\therefore \angle EGB + \angle BGH = 180°$ [Linear Pair Axiom

Also $\angle EGB =$ Corresp. $\angle GHD$ [Corresponding Angles Axiom

$\therefore \angle GHD + \angle BGH = 180°$ [By substitution

i.e., $\angle BGH + \angle GHD = 180°$ ...(1)

Again ray GA stands on line EF.

$\therefore \angle EGA + \angle AGH = 180°$ [Linear Pair Axiom

Also $\angle EGA = \angle GHC$ [Corresponding Angles Axiom

$\therefore \angle AGH + \angle GHC = 180°$ [By substitution

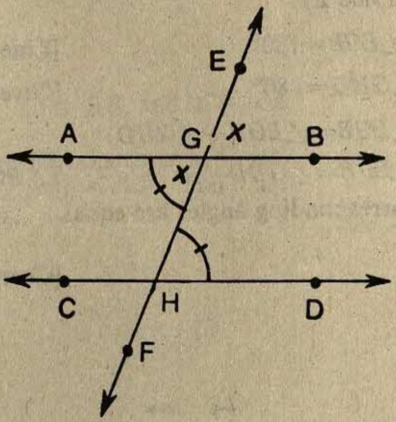
Hence $\angle BGH + \angle GHD = 180°$ and $\angle AGH + \angle GHC = 180°$.

We shall now prove the converse of these theorems.

THEOREM—6

(*Converse of Theorem 4*)

If a transversal intersects two lines in such a way that a pair of alternate interior angles are equal, then the two lines are parallel.



Given : A transversal, $\overleftrightarrow{EF}$ cutting two lines AB and CD at G and H such that $\angle AGH =$ alternate $\angle GHD$.

To prove : Lines AB and CD are parallel.

Proof : $\angle AGH = \angle EGB$ [Vertically opposite angles

$\angle AGH = \angle GHD$ [Given

$\therefore \angle EGB = \angle GHD$

Thus, a pair of corresponding angles, $\angle EGB$ and $\angle GHD$ are equal.

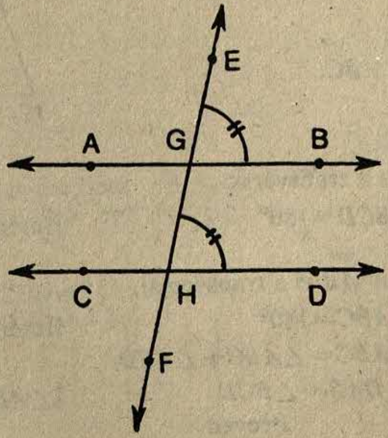
$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ [Corresponding Angles Axiom

Proved

THEOREM—7

(*Converse of Theorem 5*)

If a transversal intersects two lines in such a way that a pair of interior angles on the same side of the tranversal are supplementary, then the two lines are parallel. [*C.B.S.E. 1987, Delhi*]



Given : A transversal, EF cutting two other lines AB and CD at G and H such that $\angle BGH + \angle GHD = 180°$.

To prove : Lines AB and CD are parallel.

Proof : Ray GB stands on line EF.

$\therefore \quad \angle BGH + \angle EGB = 180°$ [Linear Pair Axiom

$\angle BGH + \angle GHD = 180°$ [Given

$\therefore \quad \angle BGH + \angle EGB = \angle BGH + \angle GHD$

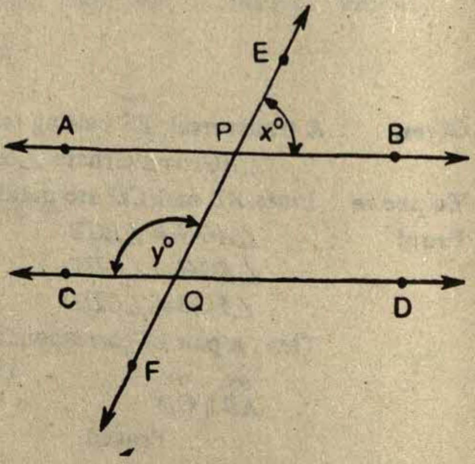
Then $\angle EGB = \angle GHD$ [$\angle BGH$ is common to both sides.

Thus, a pair of corresponding angles are equal.

$\therefore \quad \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ [Corresponding Angles Axion.

Proved

Example 1. In the given figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\angle x = 65°$. Find the magnitude of $\angle y$.



Solution. Line $AB \parallel$ line CD, and line EF is their transversal.

$\therefore \quad \angle x = \angle PQD$ [Corresponding Angles

But $\angle y + \angle PQD = 180°$ [Linear Pair Axiom

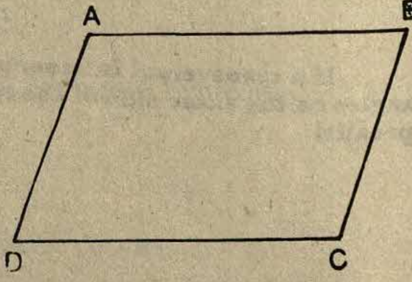
$\therefore \quad \angle y + \angle x = 180°$ [$\angle PQD = \angle x$

or $\quad \angle y + 65° = 180°$ [$\angle x = 65°$ given

$\therefore \quad \angle y = 180° - 65°$ *i.e.*, $115°$

Example 2. In the given figure, $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

Prove that $\angle DAB = \angle BCD$.



Solution

Given : $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

To prove : $\angle DAB = \angle BCD$

Proof : $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{BC}$ is a transversal

$\therefore \quad \angle ABC + \angle BCD = 180°$ [Interior Angles on the same side

Again $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$ and $\overleftrightarrow{AB}$ is a transversal.

$\therefore \quad \angle DAB + \angle ABC = 180°$ [Interior Angles on the same side

So, $\quad \angle DAB + \angle ABC = \angle ABC + \angle BCD$

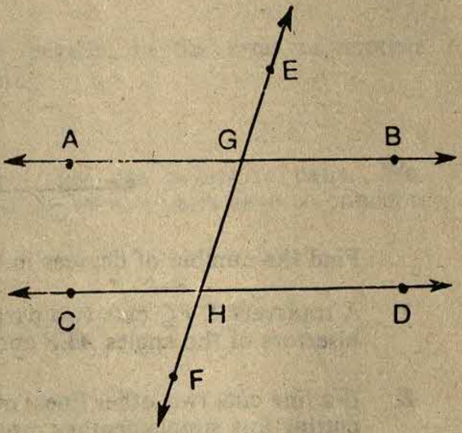
$\therefore \quad \angle DAB = \angle BCD$ [$\angle ABC$ is common

Proved

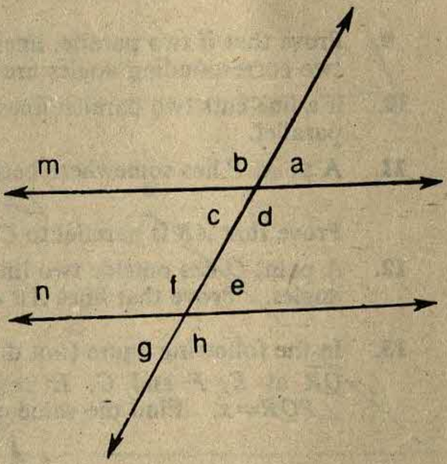
EXERCISE—12 (*c*)

(Section—A)

1. In the given figure, lines *AB* and *CD* are parallel. Transversal $\overleftrightarrow{EF}$ cuts them at *G* and *H* respectively. If $\angle AGE = 110°$, find the magnitude of $\angle GHD$.



2. In the given figure, if $\angle b = 120°$ and $\angle e = 60°$, show that lines *m* and *n* are parallel.



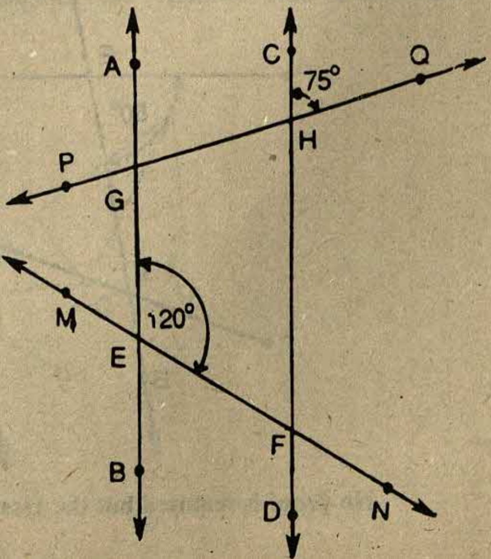
3. Prove that if two lines are perpendicular to the same line, they are parallel to each other. [*C.B.S.E., 1978* (*A.I.*)]

4. Prove that lines which are parallel to the same straight line are parallel to one other

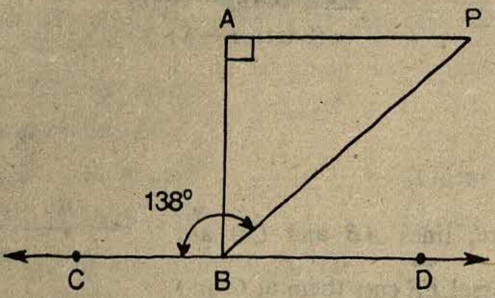
(Section—B)

5. In the accompanying figure lines *AB* and *CD* are parallel, lines *MN* and *PQ* cut them.

   If $\angle GEF = 120°$, $\angle CHQ = 75°$, find the magnitude of angles at *G* and *F*.



6. In the figure given below $\overline{AB}$ is perpendicular to $\overleftrightarrow{CD}$; $\overline{AP} \parallel \overline{CD}$, $\angle CBP = 138°$.



Find the number of degrees in (*i*) $\angle ABP$, (*ii*) $\angle APB$.

7. A transversal, $\overleftrightarrow{PQ}$ cuts two parallel lines AB and CD at E and F respectively. The bisectors of the angles AEF and CFE meet in G. Prove that $\angle EGF$ is a right angle. [*C.B.S.E., 1983* (*Delhi*)]

8. If a line cuts two other lines, making a pair of exterior angles on the same side of the cutting line supplementary, prove that the lines are parallel.

(Section—C)

9. Prove that if two parallel lines are intersected by a transversal, then bisectors of any two corresponding angles are parallel. [*C.B.S.E., 1984* (*A.I.*)]

10. If a line cuts two parallel lines, prove that the bisectors of a pair of alternate angles are parallel. [*C.B.S.E., 1984* (*A.I.*)]

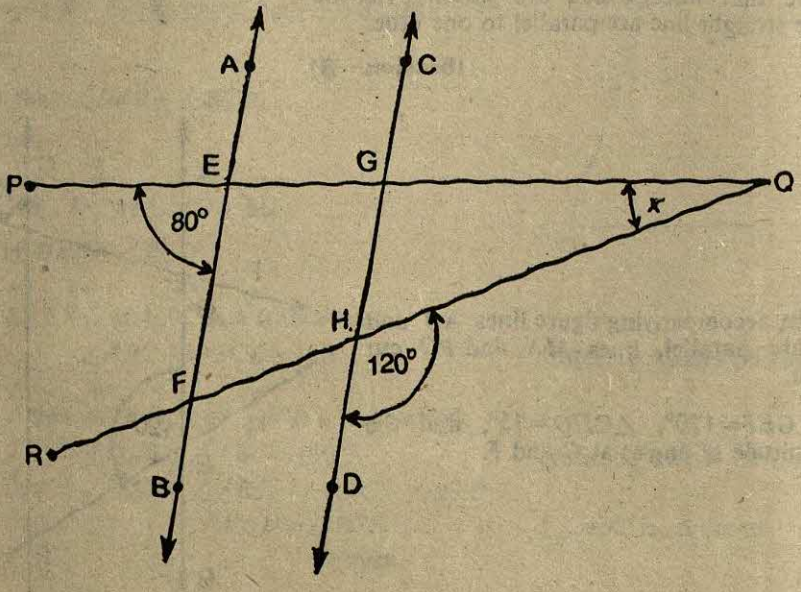
11. A point P lies somewhere between two lines AB and CD so that

$$\angle APC = \angle BAP + \angle DCP.$$

Prove that AB is parallel to CD.

12. A point Q lies outside two lines AB and CD so that $\angle ABQ + \angle BQD + \angle QDC = 4$ right angles. Prove that lines AB and CD are parallel.

13. In the following figure (not drawn to scale) $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and they cut $\overline{PQ}$ and $\overline{QR}$ at E, F and G, H respectively. Given that $\angle PEB = 80°$, $\angle QHD = 120°$ and $\angle PQR = x$. Find the value of x.



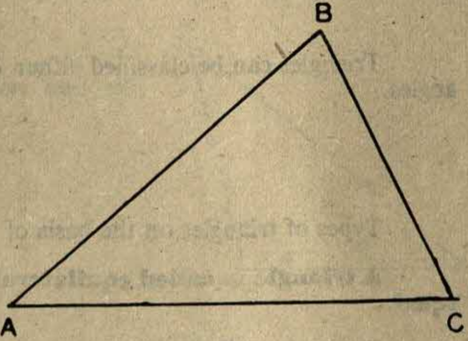
No proof is required but the essential steps of working must be given.

14 *AB* and *CD* are two parallel lines and *P* is any point in line *AB*. Through *P*, lines *PM* and *PN* are drawn cutting *CD* in *E* and *F* and $\angle EPF = 18°$. If $\angle$s *CPE, CEM* and *NFD* are in the ratio of 3 : 4 : 5, find each angle.

15. Prove that if the arms of one angle are respectively parallel to the arms of another angle, the two angles are either equal or supplementary.

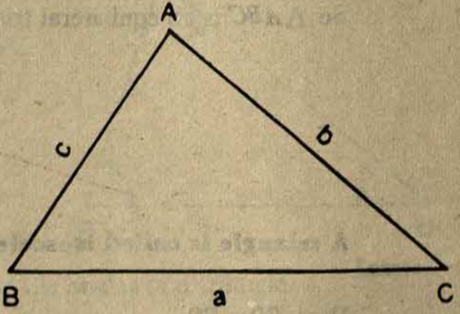
12.4. TRIANGLE AND ITS ANGLES

In a plane take three non-collinear points *A*, *B* and *C*. Join the points in pairs. We, thus, get a closed figure formed by three segments which when taken in pairs have a common end point. This figure is called a **triangle**. We use the symbol △ for denoting a triangle.



The three non-collinear points *A*, *B* and *C* are called the **vertices** of the triangle. The three segments *AB*, *BC* and *CA* are called the **sides** of the triangle. Any two sides of the triangle determine an **angle** of the triangle. One angle is formed by segments *CA* and *CB*, a second by segments *BA* and *BC* and a third by segments *AB* and *AC*.

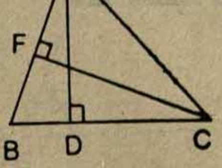
The three angles together with the three sides of the triangle are called *six parts or elements of the triangle*.

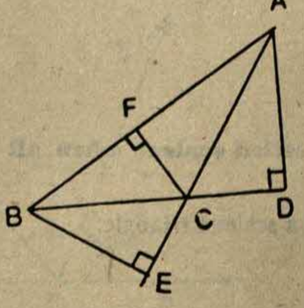


If the triangle *ABC* is made to stand on the side *BC*, then *BC* is its **base** and *A* is the vertex opposite to it.

The sides opposite to the vertices *A*, *B*, *C* are denoted by *a*, *b*, *c* respectively. Thus, a triangle has three sides and three angles, which are its six **elements** or **parts**.

The perpendicular drawn from a vertex of a triangle to the opposite side is called its **altitude**.



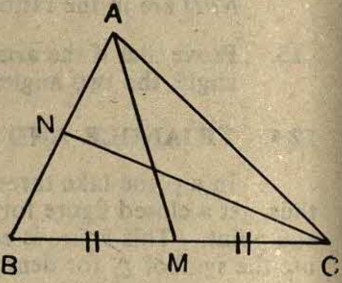


A triangle has three altitudes.

The segment joining the vertex of a triangle to the middle point of the opposite side is called a **median.**

Here ***M*** is the mid-point of the side *BC* opposite to *A* and so segment *AM* is a median.

A triangle has three medians.



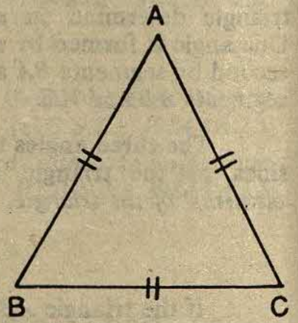
Triangles can be classified either on the basis of their sides or on the basis of their angles.

Types of triangles on the basis of sides:

A triangle is called equilateral when all its sides are equal.

Here *AB*=*BC*=*CA*.

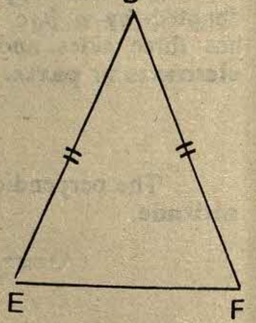
So $\triangle ABC$ is an equilateral triangle.



A triangle is called isosceles, when two of its sides are equal.

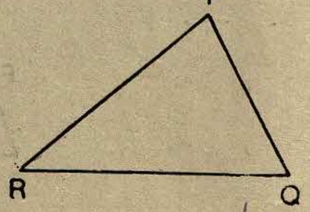
Here *ED*=*FD*.

So $\triangle ABC$ is an isosceles triangle.



A triangle is called scalene when all its sides are unequal.

Here $\triangle PQR$ is a scalene triangle.



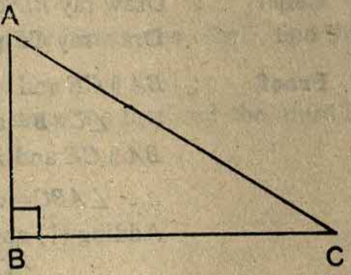
Types of triangles on the basis of angles :

A triangle is said to be right-angled, when one of its angle is a right angle.

In a right-angled triangle the side opposite to the right angle is called the **hypotenuse**.

Here $\angle ABC = 90^\circ$.

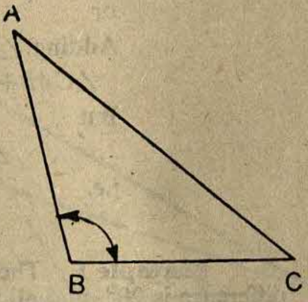
So $\triangle ABC$ is a right-angled triangle. Then AC is the hypotenuse.



A triangle is said to be obtuse-angled, when one of its angle is obtuse.

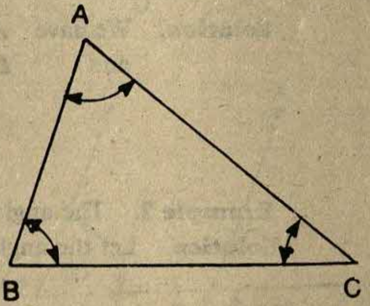
Here $\angle ABC$ is an obtuse angle.

So $\triangle ABC$ is an obtuse-angled triangle.



A triangle is said to be acute-angled when each of its angles is acute.

Here $\triangle ABC$ is an acute-angled triangle.

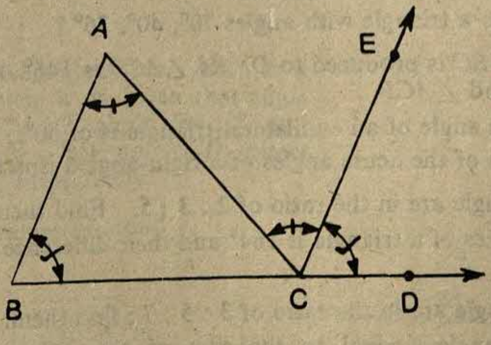


We shall, now deduce an important property about the angles of a triangle.

THEOREM—8

The sum of the three angles of a triangle is equal to two right angles.

[*C.B.S.E., 1981* (*Delhi*)]



Given : $\triangle ABC$.

To prove : $\angle A + \angle B + \angle C = 180^\circ$.

Const. : Draw ray BD containing segment BC.
Draw ray CE parallel to segment BA.

Proof : $BA \parallel CE$ and AC cuts them.

$\therefore$ $\angle CAB =$ alternate interior $\angle ACE$ ...(1)

$BA \parallel CE$ and BD cuts them.

$\therefore$ $\angle ABC =$ corresp. $\angle ECD$. .. (2)

Adding (1) and (2), we have

$$\angle CAB + \angle ABC = \angle ACE + \angle ECD$$

or $$\angle CAB + \angle ABC = \angle ACD.$$

Adding $\angle BCA$ to both sides, we get

$$\angle CAB + \angle ABC + \angle BCA = \angle ACD + \angle BCA.$$

But $\angle ACD + \angle BCA = 180^{\circ}$ [Linear Pair Axiom

$\therefore$ $\angle CAB + \angle ABC + \angle BCA = 180^{\circ}$

i.e. $\angle A + \angle B + \angle C = 180^{\circ}$

Proved.

Example 1. The sum of the angles at the base of a triangle is 150°, and their difference is 20°, find all the angles.

Solution. We have $B + C = 150^{\circ}$
and $B - C = 20^{\circ}$.
$\therefore$ $2B = 170^{\circ}$ or $B = 85^{\circ}$.
$\therefore$ $C = 150^{\circ} - 85^{\circ} = 65^{\circ}$
$\therefore$ $A = 180^{\circ} - 150^{\circ} = 30^{\circ}$.

Example 2. The angles of a triangle are in the ratio of 5 : 4 : 3 : find them.

Solution. Let the angles be denoted by $5A$, $4A$, $3A$.

$\therefore$ $5A + 4A + 3A = 180^{\circ}$
or $12A = 180^{\circ}$ or $A = 15^{\circ}$.

$\therefore$ the angles are 75°, 60° and 45°.

EXERCISE—12 (*d*)

(Section—A)

1. In $\triangle ABC$, $\angle A = 103^{\circ}$, $\angle B = 42^{\circ}$, find $\angle C$.

2. Is it possible to have a triangle with angles 70°, 40°, 36° ?

3. One side BC of $\triangle ABC$ is produced to D. If $\angle ACD$ is 146° and $\angle ABC$ is 68°, find the values of $\angle A$ and $\angle ACB$.

4. (*i*) Prove that each angle of an equilateral triangle is of 60°.
(*ii*) Prove that each of the acute angles of a right-angled isosceles triangle is of 45°.

5. The angles of a triangle are in the ratio of 2 : 3 : 5. Find them.

6. The sum of two angles of a triangle is 164° and their difference is 6° ; find all the angles of the triangle.

7. The angles of a triangle are in the ratio of 3 : 5 : 7 ; find them.

8. If one angle of a triangle is equal to the sum of the other two angles, show that the triangle is right-angled.

(Section—B)

9. One of the acute angles of a right-angled triangle is four times the other; find the number of degrees it contains.

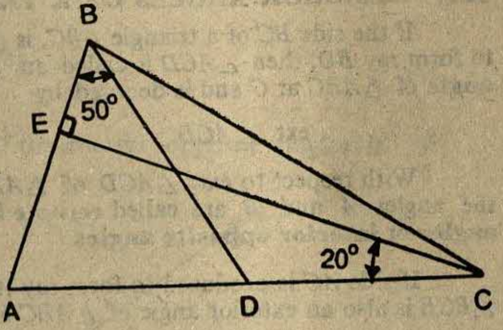
10. Find the three angles of a triangle, if the second is three times the first and the third is twice the second.

11. In a $\triangle ABC$ rt. $\angle d$ at A, $\overline{AD} \perp \overline{BC}$, prove that

$$\angle BAD = \angle C \quad \text{and} \quad \angle CAD = \angle B.$$

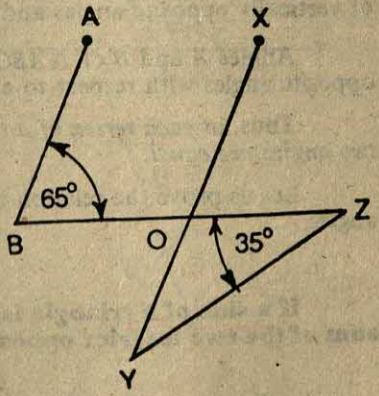
12. In the figure given alongside, $\overline{CE}$ is perpendicular to $\overline{AB}$, $\angle ACE = 20°$ and $\angle ABD = 50°$.

Find the measure of $\angle BDA$.



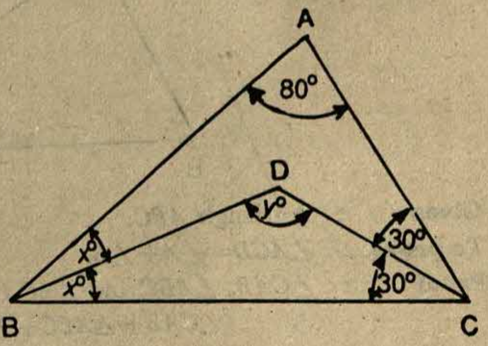
13. In the adjacent figure, $\overline{AB} \parallel \overline{XY}$, $\angle ABZ = 65°$, and $\angle BZY = 35°$.

Calculate $\angle XYZ$.



14. In the figure given alongside, ABC is a triangle in which it is given that angle $A = 80°$, angle $C = 60°$, angle $B = 2x°$ and angle $BDC = y°$. BD and CD bisect angles B and C respectively.

Calculate the values of x and y.



(Section—C)

15. If in a triangle *ABC*, the bisectors of the angles *ABC* and *ACB* meet at *P*, prove that
$$\angle BPC = 90^\circ + \frac{1}{2}\angle A.$$
[*C.B.S.E 1984* (*A.I.*)]

16. In a $\triangle ABC$, the sides *AB* and *AC* are produced and the bisectors of the exterior angles so formed meet in *O*.

Prove that $\angle BOC = 90^\circ - \frac{1}{2}\angle A$.

17. The side *BC* of a triangle *ABC* is produced to *D* ; bisectors of the angles *ABC* and *ACD* meet at *P*, prove that
$$\angle BPC = \frac{1}{2}\angle BAC.$$

12.5. EXTERIOR ANGLES OF A TRIANGLE

If the side *BC* of a triangle *ABC*, is produced to form ray *BD*, then $\angle ACD$ is called an **exterior angle** of $\triangle ABC$ at *C* and is denoted by

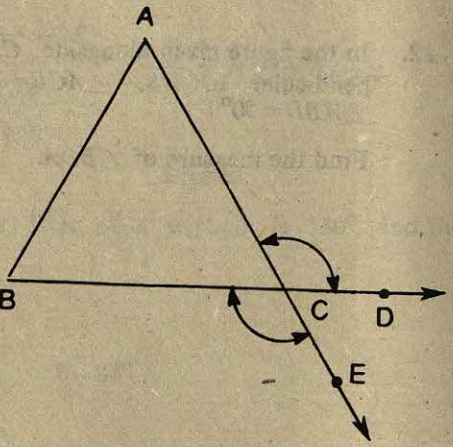
ext. $\angle ACD$.

With respect to ext. $\angle ACD$ of $\triangle ABC$ at *C*, the angles *A* and *B* are called **remote interior angles** or **interior opposite angles.**

If side *AC* is produced to form ray *AE*, then $\angle BCE$ is also an exterior angle of $\triangle ABC$ at *C*.

Observe that $\angle ACD$ and $\angle BCE$ form a pair of vertically opposite angles and they are equal.

Angles *A* and *B* of $\triangle ABC$ are also interior opposite angles with respect to ext. $\angle BCE$.

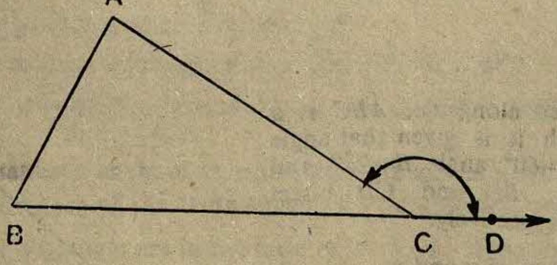


Thus, *at each vertex of a triangle, there are two exterior angles of the triangle and these two angles are equal.*

Let us prove the relation between an exterior angle of a triangle and its interior opposite angles.

THEOREM—9

If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.



Given : A triangle *ABC*.

To prove : $\angle ACD = \angle A + \angle B$.

Proof : $\angle CAB$, $\angle ABC$ and $\angle BCA$ are the angles of $\triangle ABC$.

$\therefore \quad \angle CAB + \angle ABC + \angle BCA = 180^\circ$.

Also $\angle BCA + \angle ACD = 180°$. (Linear Pair Axiom)

Then $\angle CAB + \angle ABC + \angle BCA = \angle BCA + \angle ACD$

$\therefore$ $\angle CAB + \angle ABC = \angle ACD$

i.e., $\angle ACD = \angle A + \angle B$.

Proved

From this result, it is clear that

$$\angle ACD > \angle A \quad \text{and} \quad \angle ACD > \angle B.$$

Thus, *an exterior angle of a triangle is greater than either of the interior opposite angle.*

Such a result which is an easy consequence of a theorem is called a **corollary** of that theorem.

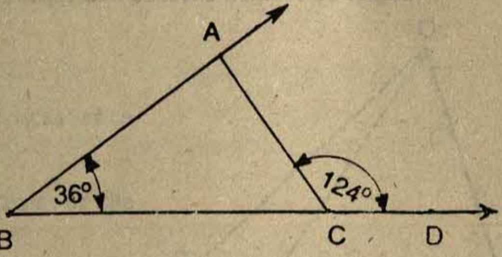
EXERCISE—12 (*e*)

(Section—A)

1. An exterior angle of a triangle is 115°, and one of the interior opposite angles is 35°. Find the other two angles of the triangle.

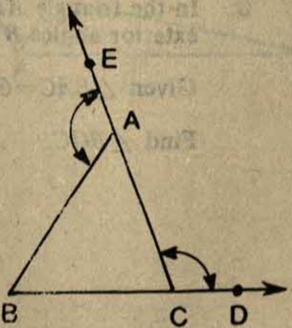
2. In the given figure, $\angle ACD = 124°$ and $\angle ABC = 36°$.

   Find the measure of $\angle BAC$.

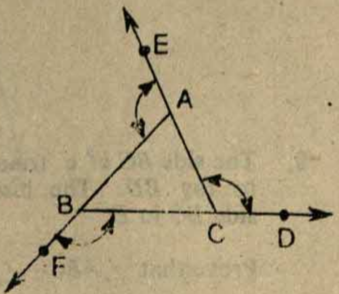


3. *ABC* is a triangle in which *BC* is produced to *D* ; *CA* is produced to *E* $\angle DCA = 108°$ and $\angle EAB = 124°$.

   Calculate $\angle ABC$.

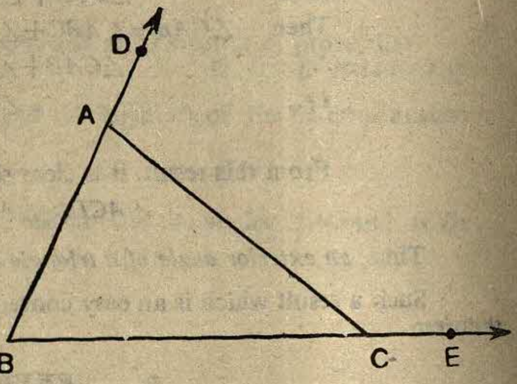


4. If the sides of a triangle are produced in order, the sum of the exterior angles so formed is equal to four right angles. Prove it.



5. The sides BA and BC of the $\triangle ABC$ are produced to D and E respectively.

Prove that $\angle DAC + \angle ECA > 180°$.



(Section—B)

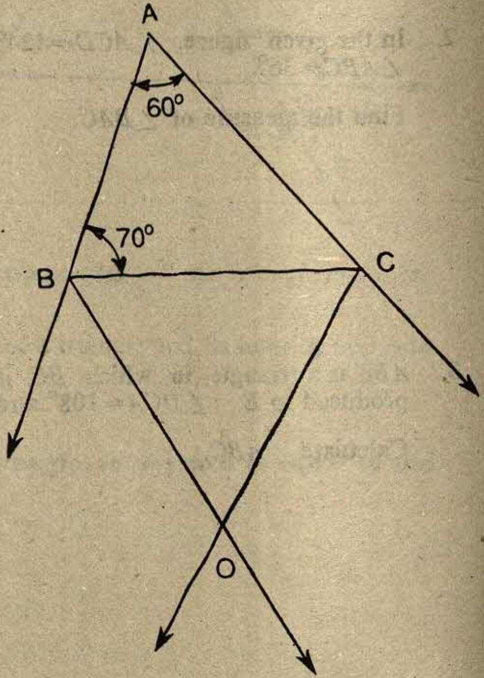
6. If the base of any triangle is produced both ways, show that the sum of the two exterior angles minus the vertical angle is equal to two right angles.

7. If the base of a triangle is produced both ways, prove that the sum of the two exterior angles so formed exceeds two right angles by the angle at the vertex.

8. In the triangle ABC, the bisectors of the exterior angles B and C meet at O.

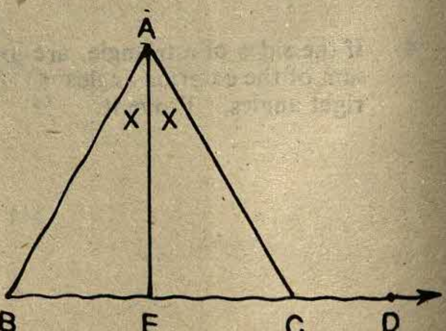
Given $\angle BAC = 60°$, and $\angle ABC = 70°$.

Find $\angle BOC$.



9. The side BC of a triangle ABC is produced to ray BD. The bisector of $\angle BAC$ meets side BC in E.

Prove that $\angle ABC + \angle ACD = 2\angle AEC$.

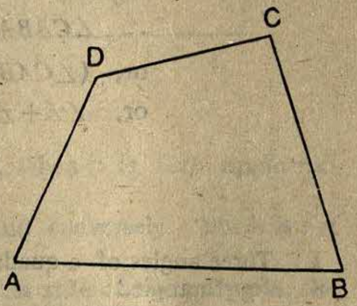


Section—C

10. In a triangle *ABC*, the sides *AB* and *AC* are produced and the bisectors of the exterior angles so formed meet at the point *O*.

Prove that $\angle BOC = 90^\circ - \frac{1}{2}\angle A$ [*C.B.S.E., 1981* (*A.I.*)]

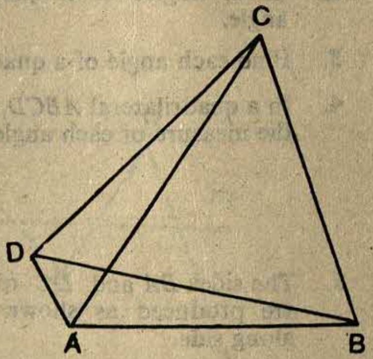
12.6. QUADRILATERAL



Here *A, B, C, D* are four *distinct* points in a plane such that *no three of them are collinear* and line segments *AB, BC, CD* and *DA do not intersect* except at their end points.

The plane figure so formed is called the **quadrilateral** with vertices *A, B, C* and *D*. It is denoted by quad. *ABCD*

The four segments *AB, BC, CD* and *DA* are called the **sides,** the four angles, $\angle A$, $\angle B$, $\angle C$ and $\angle D$ are the **angles** of the quadrilateral.



The segment which joins the opposite vertices of a quadrilateral is called a **diagonal.**

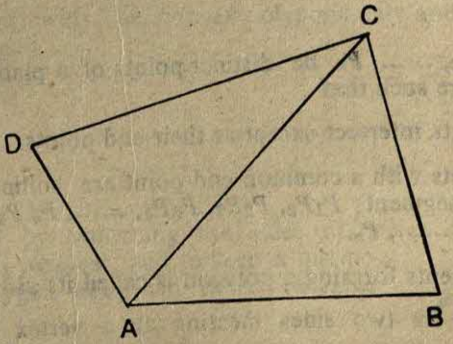
Here segments *AC* and *BD* are the diagonals of the quadrilateral *ABCD.*

A quadrilateral has *two* diagonals.

Let us now prove the angle sum property of a quadrilateral.

THEOREM—10

The sum of the four angles of a quadrilateral is 360°.



Given : Quad. *ABCD.*

To prove : $\angle A + \angle B + \angle C + \angle D = 360^\circ$.

Construction : Draw diagonal AC.

Proof : In $\triangle ABC$,

$\angle CAB+\angle B+\angle BCA=180^\circ$ [Angle sum Property of a triangle

In $\triangle ADC$,

$\angle CAD+\angle D+\angle DCA=180^\circ$ [Angle sum Property of a triangle

$\therefore\ \angle CAB+\angle B+\angle BCA+\angle CAD+\angle D+\angle CDA=180^\circ+180^\circ$

or, $(\angle CAB+\angle CAD)+\angle B+(BCA+\angle DCA)+\angle D=360^\circ$

or, $\angle A+\angle B+\angle C+\angle D=360^\circ$ Proved.

EXERCISE—12 (*f*)

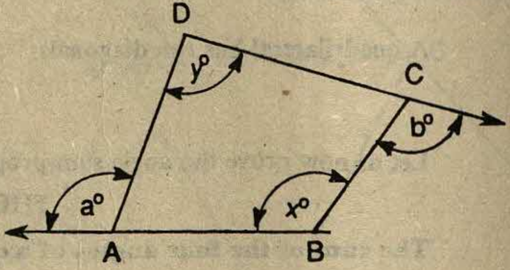
(Section—A)

1. Three angles of a quadrilateral are respectively equal to 110°, 50° and 40°. Find its fourth angle.
2. Three angles of a quadrilateral are respectively 100°, 55° and 82°. Find the fourth angle.
3. Find each angle of a quadrilateral, if its angles are in the ratio of 3 : 4 : 5 : 6.
4. In a quadrilateral $ABCD$, the angles A, B, C and D are in the ratio 1 : 2 : 3 : 4. Find the measure of each angle of the quadrilateral.

(Section—B)

5. The sides BA and DC of quad. $ABCD$ are produced as shown in the figure along side.

   Prove that $\angle a+\angle b=\angle x+\angle y$.



12.7. POLYGONS

For $n > 3$, let $P_1, P_2, P_3, \ldots\ldots, P_n$ be distinct points of a plane. If the n line segments $P_1P_2, P_2P_3, P_3P_4, \ldots\ldots P_n P_1$ are such that

(*i*) no two line segments intersect except at their end points.

(*ii*) no two line segments with a common end point are collinear, then the plane figure formed by n line segments $P_1P_2, P_2P_3, P_3P_4, \ldots\ldots, P_n P_1$ is called a **polygon** with vertices $P_1, P_2, P_3, \ldots\ldots, P_n$.

Each of the n line segments forming a polygon is called its **side**.

The angle determined by two sides meeting at a vertex is called an **angle** of the polygon.

Note that a triangle is a polygon of three sides and a quadrilateral is a polygon of four sides.

With reference to the *number of sides* the polygons are classified as follows :

No. of sides	*Name*	*No. of sides*	*Name*
5	Pentagon	8	Octagon
6	Hexagon	9	Nonagon
7	Heptagon	10	Decagon

A ploygon is said to be

equilateral, when all its sides are equal,

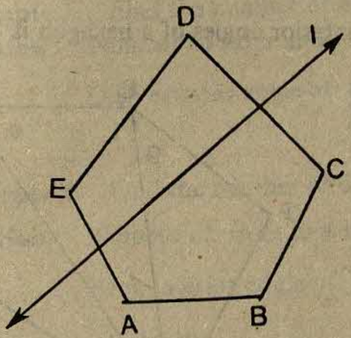
equiangular, when all its angles are equal,

regular, when all its sides and all its angles are equal, *i.e.*, when it is *both equilateral and equiangular.*

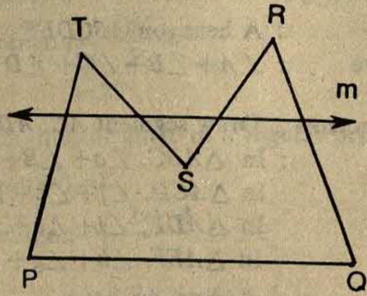
Note : If a triangle is equilateral, it is also equiangular and conversely. **But it is not** so in the case of polygons.

A polygon $P_1 P_2 P_3 \ldots \ldots P_n$ is called convex, if for each side of the polygon, the line containing that side has all other vertices on the same side of it.

Observe the following polygons :



Convex polygon



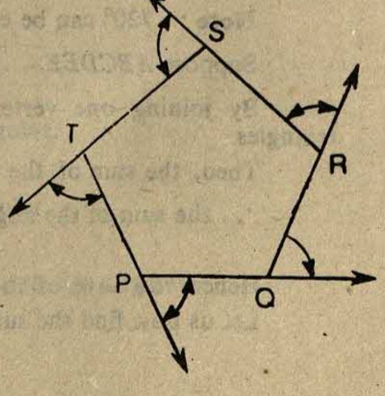
Concave polygon

A convex polygon can be cut by a line in *two and only two points.* (See above).

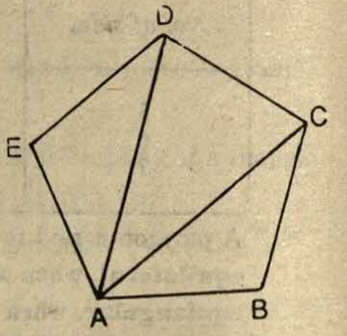
A concave polygon can be cut by a line in *more than two points.* (See above).

We shall be dealing with the convex polygons only and as such a polygon wherever used will be a convex polygon.

The angles formed by producing the sides of a polygon are called **exterior angles** ; and to keep a distinction the angles of the polygon **are** then called **interior angles.**



Example 1 : Prove that the sum of the interior angles of a pentagon is 540°.



Given : A pentagon *ABCDE*.

To prove : $\angle A+\angle B+\angle C+\angle D+\angle E=540^\circ$.

Construction : Draw segments *AC* and *AD* joining vertex *A* to *C* and *D* respectively.

Proof : The pentagon is divided into triangles *ABC*, *ACD* and *ADE*.

: In $\triangle ABC$, $\angle CAB+\angle CBA+\angle BCA$ $=2$ rt. angles.

: In $\triangle ACD$, $\angle DAC+ADC+\angle DCA$ $=2$ rt. angles.

In $\triangle ADE$, $\angle DAE+\angle AED+\angle EDA$ $=2$ rt. angles.

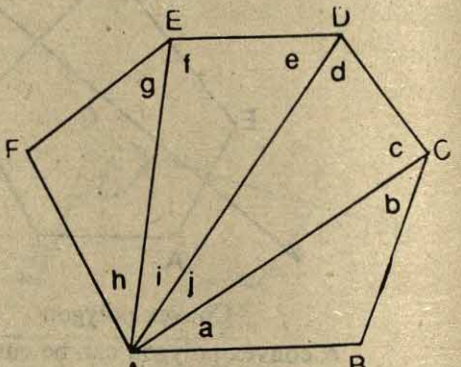
Adding we get,

$$(\angle CAB+\angle DAC+\angle DAE)+\angle CBA+(\angle BCA+\angle DCA)+(\angle ADC+\angle EDA)+\angle AED=6 \text{ rt. angles.}$$

$$\angle EAB+\angle CBA+\angle DCB+\angle CDE+\angle AED=6 \text{ rt. angles.}$$

Note : 540° can be expressed as $(2\times 5-4)$ rt. angles.

Example 2 : Prove that the sum of all the interior angles of a hexagon is 720°.



Given : A hexagon *ABCDEF*

To prove : $\angle A+\angle B+\angle C+\angle D+\angle E+\angle F.$ $=720^\circ$.

Construction : Draw segment *AC*, *AD* and *AE*.

Proof : In $\triangle ABC$, $\angle a+\angle B+\angle b=180^\circ$
In $\triangle ACD$, $\angle j+\angle c+\angle d=180^\circ$
In $\triangle ADE$, $\angle i+\angle e+\angle f=180^\circ$
In $\triangle AEF$, $\angle h+\angle g+\angle F=180^\circ$

Adding, we have,

$$\angle a+\angle B+\angle b+\angle j+\angle c+\angle d+\angle i+\angle e+\angle f+\angle h+\angle g+\angle F=720^\circ.$$

or $(\angle a+\angle j+\angle i+\angle h)+\angle B+(\angle b+\angle c)+(\angle d+\angle e)+(\angle f+\angle g)+\angle F=720^\circ$.

i.e., $\angle A+\angle B+\angle C+\angle D+\angle E+\angle F=720^\circ$.

Note : 720° can be expressed as $(2\times 6-4)$ rt. angles.

Suppose *ABCDEE*......is a polygan of *n* sides ($n \geqslant 3$).

By joining one vertex to the remaining vertices, the polygon is divided into $(n-2)$ triangles.

Then, the sum of the angles of one triangle $=2$ rt. $\angle s$

$\therefore$ the sum of the angles of $(n-2)$ triangles $=(n-2)\,.\,2$ rt. $\angle s$
$=(2n-4)$ rt. $\angle s$

Hence, **the sum of the interior angles of an *n*-sided polygon** $=(2n-4)$ **rt** $\angle$**s**.

Let us now find the sum of the exterior angles of a polygon of *n* sides

At each vertex of the polygon, there is an interior angle and an exterior angle whose sum is 2 rt. $\angle s$.

There are n vertices in an n-sided polygon.

$\therefore$ the sum of all the interior angles+the sum of all the exterior angles$=2n$ rt. $\angle s$

But the sum of all the interior angles of the polygon$=(2n-4)$ rt. $\angle s$

$\therefore$ the sum of all the exterior angles of the polygon$=2n$ rt. $\angle s-(2n-4)$ rt. $\angle s$

$=4$ rt. $\angle s$.

Hence, the sum of the exterior angles of a polygon is 4 right angles

Learn the following corollaries :

Corollary 1. Each interior angle of a regular polygon of n sides is equal to

$$\left(\frac{2n-4}{n}\right) \text{rt. } \angle s \text{ or } \left(\frac{2n-4}{n}\right)\times 90°.$$

Corollary 2. Each exterior angle of a regular polygon of n sides is equal to $\frac{4\text{rt. } \angle s}{n}$ or $\left(\frac{360}{n}\right)^{\circ}$.

Example 1. Find the number of degrees in each exterior angle of a regular pentagon.

Solution : Number of sides in a regular pentagon $(n)=5$.

Each exterior angle of a regular pentagon$=\frac{360°}{5}$

$=72°$.

Example 2. Find the number of degrees in each interior angle of a regular octagon.

Solution : Number of sides in a regular octagon $(n)=8$.

Each interior angle of a regular octagon$=\left(\frac{2n-4}{n}\right)\times 90°$

$=\left(\frac{2\times 8-4}{8}\right)\times 90°$

$=135°$.

EXERCISE—12 (g)

(Section—A)

1. A regular polygon has exterior angle of $x°$.
   Find in terms of x,
   (*a*) the measure of an interior angle of the polygon ;
   (*b*) the number of sides of the polygon.
2. Find the number of degrees in each exterior angle of a regular,
   (*a*) hexagon, (*b*) octagon, (*c*) decagon, (*d*) dodecagon.
3. Find the number of degrees in each interior angle of a regular,
   (*a*) pentagon, (*b*) hexagon, (*c*) decagon, (*d*) dodecagon.
4. Find the number of sides of a regular polygon whose each exterior angle is
   (*a*) 120°, (*b*) 20°, (*c*) 24°, (*d*) 40°.

5. Each exterior angle of a regular polygon is 36°. Find the number of sides of the polygon.

(Section—B)

6. Find the number of sides of a polygon each of whose interior angles is
(*a*) 144°, (*b*) 150°, (*c*) 160°, (*d*) 162°.

7. Find the number of sides of a regular polygon if each of its interior angles is 135°.

8. *ABCDE* is a regular pentagon. The bisector of $\angle A$ of the pentagon meets the side *CD* in *M*. Show that $\angle AMC = 90°$.

9. The angles of a pentagon are $x°$, $(x-10)°$, $(x+20)°$, $(2x-44)°$ and $(2x-70)°$. Calculate x.

10. How many sides has a polygon, if the sum of all the interior angles is
(*a*) 12 rt $\angle s$, (*b*) 18 rt. $\angle s$, (*c*) 26 rt. $\angle s$.

11. The sum of interior angles of a polygon is 1440°. How many sides has this polygon ?

(Section—C)

12. The exterior angle of a regular polygon is one-third of its interior angle. How many sides has the polygon ?

13. The sum of the interior angles of a polygon is three times the sum of its exterior angles. How many sides has it got ?

14. If the angles of a pentagon, taken in order are in the ratio 4 : 8 : 6 : 4 : 5, prove that two pairs of sides are parallel.

REVIEW EXERCISE—XI

(Section— A)

1. Fill in the blanks to make the following statements true :

(*a*) If a ray stands on a line, then the two adjacent angles so formed are... [*C.B.S.E., 1979 (A.I.)*]

(*b*) If a transversal intersects two parallel lines, then the interior angles on the same side of the transversal are ..... [*C.B.S.E., 1979 (A.I.)*]

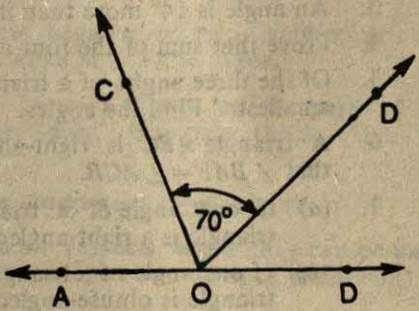
(*c*) If a transversal intersects two lines in such a way that alternate angles in any pair are equal, then the two lines are ..... [*C.B.S.E. 1980 (A.I.)*]

(*d*) Lines which are perpendicular to the same line are...... to each other. [*C.B.S.E. 1983 (A.I.)*]

(*e*) Two lines which are parallel to the same line are...... to each other. [*C.B.S.E., 1979 (Delhi)*]

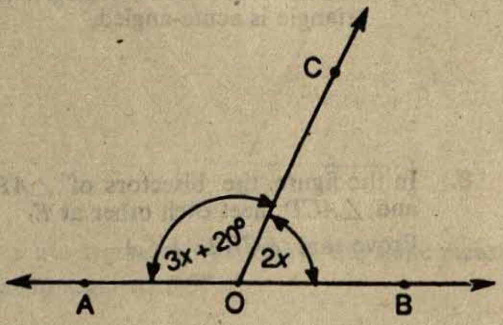
(*f*) In the figure, if $\angle DOC = 70°$, then $\angle AOC + \angle BOD =$......

[*C.B.S.E., 1980 (Delhi)*]



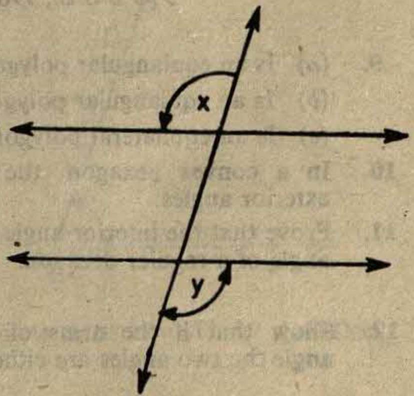
(*g*) In the given figure, if *AOB* is a line, then $x =$......

[*C.B.S.E., 1984 (A.I.)*]



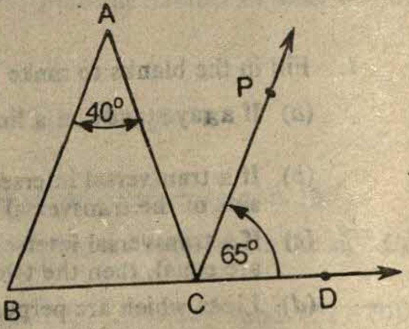
(*h*) In the figure, if $\angle x = \angle y$, then lines *AB* and *CD* are......

[*C.B.S.E., 1982 (A.I.)*]



(*i*) In the figure, if ray CP is parallel to segment AB, $\angle BAC = 40^\circ$, and $\angle PCD = 65^\circ$, then $\angle ACB = \ldots\ldots$

[*C B.S.E., 1980 (Delhi)*]



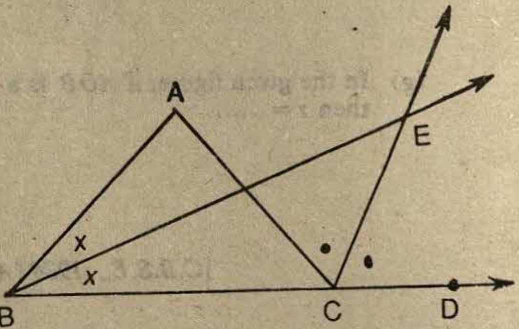
2. How many diagonals has (*a*) a pentagon, (*b*) a hexagon ?

(Section—B)

3. An angle is 14° more than its complement. What is its measure ?
4. Prove that sum of the four angles of a quadrilateral is 360°.
5. Of the three angles of a triangle, one is twice the smallest and another is three times the smallest. Find the angles.
6. A triangle ABC is right-angled at A. AP is drawn perpendicular to side BC. Prove that $\angle BAP = \angle ACB$.
7. (*a*) If one angle of a triangle is equal to the sum of the other two, show that the triangle is a right-angled.
   (*b*) If one angle of a triangle is greater than the sum of the other two, show that the triangle is obtuse-angled.
   (*c*) If one angle of a triangle is less than the sum of the other two, show that the triangle is acute-angled.

8. In the figure, the bisectors of $\angle ABC$ and $\angle ACD$ meet each other at E. Prove that $\angle CEB = \frac{1}{2}\angle A$

[*C B S E., 1982 (Delhi)*]



9. (*a*) Is an equiangular polygon always equilateral ?
   (*b*) Is an equiangular polygon regular ?
   (*c*) Is an equilateral polygon regular ?
10. In a convex hexagon, the sum of the interior angles is equal to twice the sum of the exterior angles.
11. Prove that the interior angle of a regular pentagon is three times as great as the exterior angle of a regular decagon.

(Section—C)

12. Show that if the arms of one angle are respectively parallel to the arms of another angle the two angles are either equal or supplementary.

□ □

13

CONGRUENCE OF TRIANGLES

13.1. CONGRUENCY

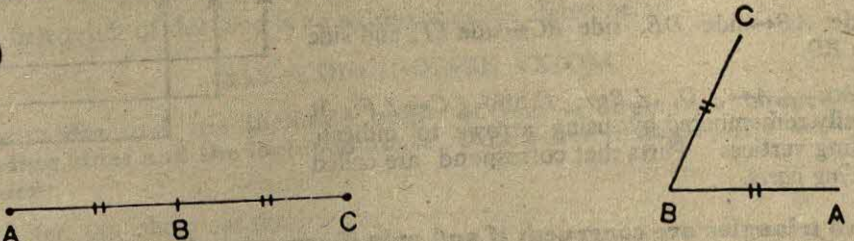
Study two photographs developed from the same negative or two pictures printed from the same block. You will see that they are exactly alike in size and shape.

Put a carbon paper between two sheets of paper. Draw a figure on the paper at the top You will get a carbon copy of the same on the other sheet. Compare them. The two figures are exactly alike in size and shape

Figures that are exactly alike in size and shape are **congruent** to each other.

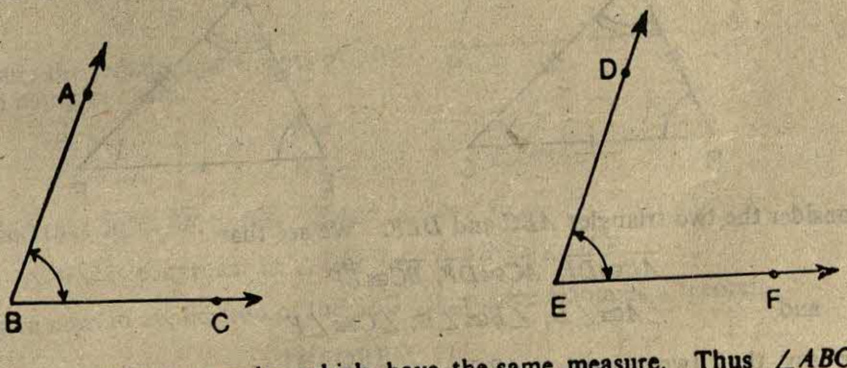
Two figures are said to be **congruent** or **equal** in all respects when they can be made to coincide with each other by superposition. (Congruency is the noun form of congruent).

(*a*)



You may recall that congruent segments are segments which have the same measure. Thus $\overline{AB} \cong \overline{BC}$ is read "Segment *AB* is congruent to segment *BC*".

(*b*)



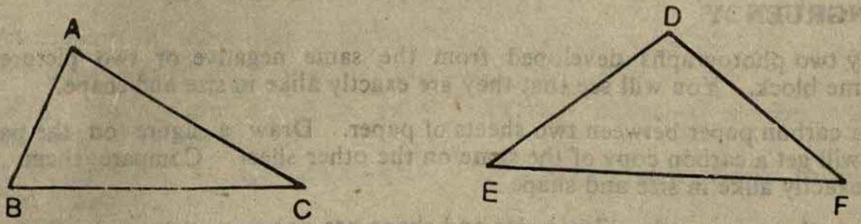
Congruent angles are angles which have the same measure. Thus $\angle ABC \cong \angle DEF$ because the measure of each is the same.

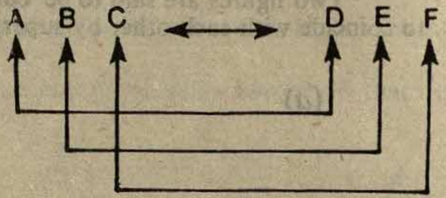
Remember that when we speak or write equal segments or equal angles we mean segments or angles whose measures are equal. The segments and angles, strictly speaking, are congruent.

13.2. CONGRUENT TRIANGLES

The shape and size of a triangle depends on its six elements, namely its three sides and three angles. We, therefore, define the congruence of two triangles in terms of the congruence of their sides and angles.

Let us consider any two triangles *ABC* and *DEF* in the same plane. Each triangle is determined by three vertices, it is possible to establish a one-to-one correspondence between the vertices of two triangles.



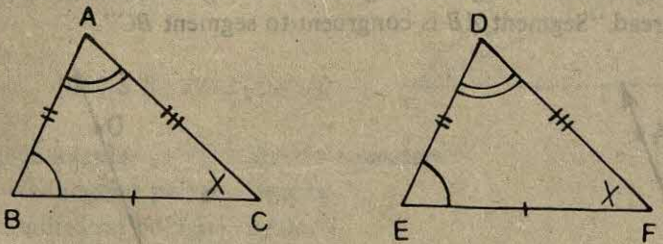


If we write $ABC \leftrightarrow DEF$, then we mean that *A* corresponds to *D*, *B* to *E*, and *C* to *F*. As a result of this we get.

Side $AB \leftrightarrow$ side DE, side $BC \leftrightarrow$ side EF, and side $CA \leftrightarrow$ side FD.

Also, $\angle A \leftrightarrow \angle D$, $\angle B \leftrightarrow \angle E$ and $\angle C \leftrightarrow \angle F$. It can be easily remembered by using arrows to indicate correspoding vertices Parts that correspond are called *corresponding parts*.

Two triangles are congruent if and only if corresponding sides are congruent and corresponding angles are congruent.



Consider the two triangles *ABC* and *DEF*. We see that

$$\overline{AB} \cong \overline{DE},\ \overline{AC} \cong \overline{DF},\ \overline{BC} \cong \overline{EF}$$

and $$\angle A \cong \angle D,\ \angle B \cong \angle E,\ \angle C \cong \angle F$$

We say that two triangles *ABC* and *DEF* are congruent. Symbolically, we write, $\triangle ABC \cong \triangle DEF$. This notation shows the correspondence $ABC \leftrightarrow DEF$, *i.e.*, $A \leftrightarrow D$, $B \leftrightarrow E$ and $C \leftrightarrow F$. We see that congruent triangles have the same size and shape. They need not be placed in similar positions in the same plane.

The definition of congruence of triangles and the above notation lead to the following results :

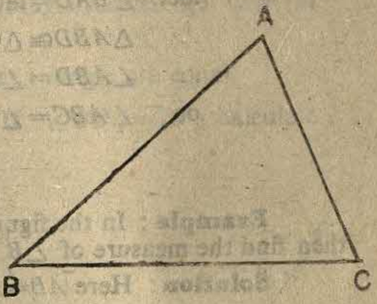
(*i*) $\triangle ABC \cong \triangle ABC$

(*ii*) If $\triangle ABC \cong \triangle DEF$, then $\triangle DEF \cong \triangle ABC$

(*iii*) If $\triangle ABC \cong \triangle DEF$, and $\triangle DEF \cong \triangle PQR$, then $\triangle ABC \cong \triangle PQR$.

The definition of congruence states that two triangles are congruent if and only if all the six elements of one triangles are congruent to corresponding six elements of the other triangle. However, it may be true that we may require fewer than six elements to be congruent *i.e.*, if certain *corresponding parts* are congruent, the remaining parts must be congruent.

Two sides of a triangle are said to *include* an angle if the vertex of the angle is the common end point of each side. Sides AB and AC include angle A. Two angles are said to *include* a side if their vertices are end points of that side. $\angle B$ and $\angle C$ include side BC.



What are the least possible conditions to ensure the congruence of two triangles ?

If three properly chosen elements out of six elements of one triangle are congruent to their corresponding elements of the other triangle, the remaining three elements are automatically congruent to their corresponding elements.

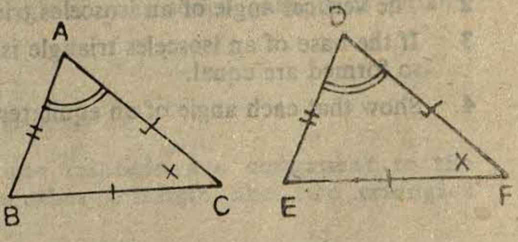
Our first result of this kind is an **axiom** which is stated below :

SAS—CONGRUENCE AXIOM

If two sides and the included angle of one triangle are congruent to the corresponding sides and the included angle of the other triangle, the two triangles are congruent.

We refer the above as the *side-angle-side* (briefly written as SAS) for congruency of two triangles.

Here $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$ and $\angle A \cong \angle D$



Measure the remaining elements of the triangles and compare them.

$BC =$ ...... ; $\angle B =$ ...... ; $\angle C =$ ......

$EF =$ ...... ; $\angle E =$ ...... ; $\angle F =$ ......

We find that $\overline{BC} \cong \overline{EF}$, $\angle B \cong \angle E$ and $\angle C \cong \angle F$.

Thus, $\triangle ABC$ is congruent to $\triangle DEF$ *i.e.*, $\triangle ABC \cong \triangle DEF$.

We give now an application of the SAS congruence axiom as a theorem.

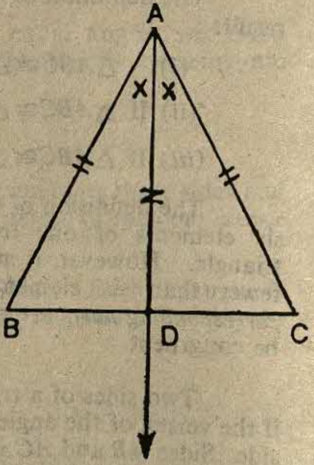
THEOREM 11

If two sides of a triangle are equal, the angles opposite to these sides are equal. [*C.B.S.E., 1983 (Delhi)*]

Given : $\triangle ABC$ in which $AC=AB$

To prove : $\angle ABC=\angle ACB$.

Const. : Draw ray AD, the bisector of $\angle BAC$, intersecting side BC in D.



Proof : In $\triangle ABD$ and $\triangle ACD$

$AB=AC$ (Given)

$AD=AD$ (Common)

incl. $\angle BAD=$ incl. $\angle CAD$ (Construction)

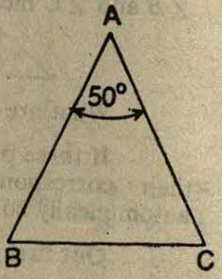
$\therefore$ $\triangle ABD \cong \triangle ACD$ (SAS, congruence Axiom)

$\therefore$ $\angle ABD=\angle ACD$ (Corresponding parts of

or $\angle ABC=\angle ACB$ congruent triangles).

Proved

Example : In the figure, if $AB=3$ cm, $AC=3$ cm and $\angle A=50°$, then find the measure of $\angle B$.



Solution : Here $AB=AC=3$ cm.

$\therefore$ Angles opposite to equal sides are equal.

i.e., $\angle ABC=\angle ACB$

The sum of the angles of $\triangle ABC$ is equal to $180°$.

$\therefore$ $\angle ABC+\angle ACB+\angle BAC=180°$

or $\angle ABC+\angle ABC+50°=180°$

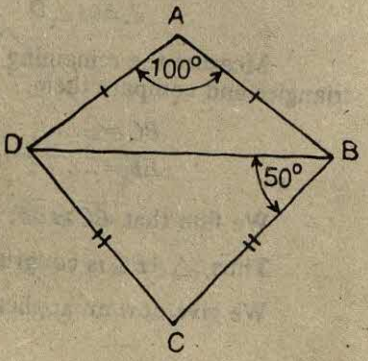
or $2\angle ABC=130°$

$\therefore$ $\angle ABC=65°$

EXERCISE—13 (*a*)

(Section—A)

1. In $\triangle ABC$, if $\angle A=100°$ and $AB=AC$, find $\angle B$ and $\angle C$.
2. The vertical angle of an isosceles triangle is $50°$. Find the base angle.
3. If the base of an isosceles triangle is produced both ways, show that the exterior angles so formed are equal.
4. Show that each angle of an equilateral triangle is $60°$.



5. In the figure, it is given that $AB=AD$, $BC=CD$, $\angle BAD=100°$, and $\angle DBC=50°$. Calculate $\angle ADC$ and $\angle BCD$.
6. Two isosceles triangles ABC and DBC are drawn on the same side of the same base BC. Prove that $\angle ABD=\angle ACD$.

7. Two isosceles triangles *ABC* and *DBC* are drawn on opposite sides of the same base *BC*. Prove that $\angle ABD = \angle ACD$.

(Section—B)

8. Each of the base angles of an isosceles triangle is (*a*) twice, (*b*) one-third of the vertical angle. Find all the angles.

9. The vertex angle of an isosceles triangle is one-fourth of a base angle. How many degrees are in each angle ?

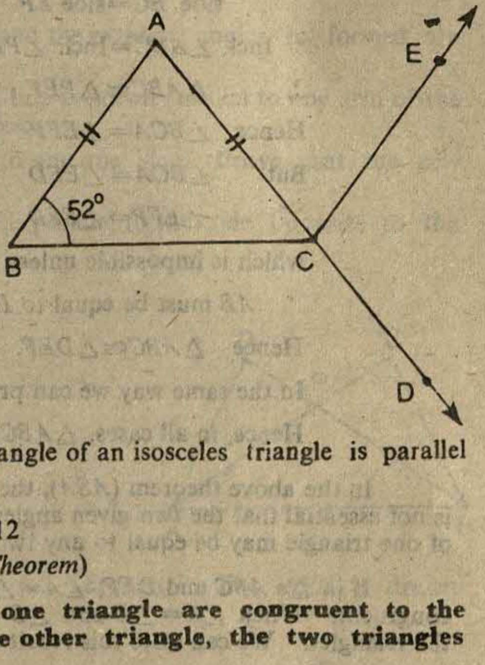
10. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

11. Prove that the medians bisecting the equal sides of an isosceles triangle are equal.

12. In a triangle *ABC*, *AD* bisects $\angle BAC$ and $AD = DC$. If $\angle BDA = 76°$, calculate $\angle ACD$ and $\angle ABD$.

(Section—C)

13. In the given figure, *ABC* is an isosceles triangle with *AB* equal to *AC*. *AC* is produced to point *D* and *CE* is drawn parallel to *BA*. If $\angle CBA = 52°$, find $\angle DCE$.



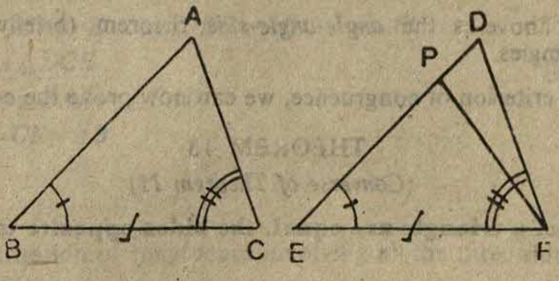
14. Prove that the external bisector of the vertical angle of an isosceles triangle is parallel to the base.

THEOREM—12

(*ASA—Congruence Theorem*)

If two angles and the included side of one triangle are congruent to the corresponding angles and the included side of the other triangle, the two triangles are congruent.

We refer the above as the *angle-side-angle* theorem (briefly written as ASA) for congruence of two triangles.



Given : In $\triangle ABC$ and $\triangle DEF$

$\angle ABC = \angle DEF$

$\angle BCA = \angle EFD$

and side $BC =$ side EF

To prove : $\triangle ABC \cong \triangle DEF$

Proof : If $AB = DE$, then $\triangle ABC$ is congruent to $\triangle DEF$ by SAS criterion and the theorem is proved.

Suppose $AB \neq DE$ and let AB be less than DE.

Take a point P on DE such that $EP = BA$. Join PF.

In $\triangle ABC$ and $\triangle PEF$

side $AB =$ side PE (Supposed)

side $BC =$ side EF (Given)

Incl. $\angle ABC =$ Incl. $\angle PEF$ (Given)

$\therefore$ $\triangle ABC \cong \triangle PEF$ (SAS-Axiom)

Hence $\angle BCA = \angle EFP$ (Corresponding parts of congruent triangles)

But $\angle BCA = \angle EFD$ (Given)

$\therefore$ $\angle EFP = \angle EFD$

which is impossible unless ray FP coincides with ray FD or P coincides with D.

$\therefore$ AB must be equal to DE.

Hence $\triangle ABC \cong \triangle DEF$.

In the same way we can prove that $\triangle ABC$ is congruent to $\triangle DEF$, when $AB > DE$.

Hence, in all cases, $\triangle ABC \cong \triangle DEF$.

In the above theorem (*ASA*), the side should be included between two angles. But it is not essential that the two given angles should be adjacent to the given side Any two angles of one triangle may be equal to any two angles of the other.

If in $\triangle$s ABC and DEF, $\angle A = \angle D$, $\angle B = \angle E$ and $BC = EF$, then also the triangles are congruent. When $\angle A = \angle D$ and $\angle B = \angle E$, then $\angle C = \angle F$, being the remaining angles of the triangles. We can state this result as under :

If any two angles and a non-included side of one triangle are equal to the corresponding angles and side of another triangle, then the two triangles are congruent.

We refer to the above as the *angle-angle-side* theorem (briefly written as *AAS*) for congruency of two triangles.

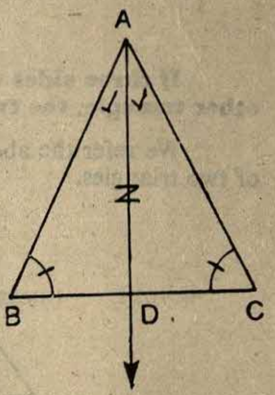
Using the *ASA* criterion of congruence, we can now prove the converse of theorem 11.

THEOREM 13

(*Converse of Theorem 11*)

If two angles of a triangle are equal, the sides opposite to them are equal.

[*C.B.S.E., 1980* (*Delhi*)]



Given : $\triangle ACB$ in which $\angle ACB = \angle ABC$.

To prove : $AB = AC$

Const. : Draw ray AD, the bisector of $\angle BAC$, intersecting BC in D.

Proof : In $\triangle ABD$ and $\triangle ACD$

$\angle ABD = \angle ACD$ (Given)

$\angle BAD = \angle CAD$ (Construction)

$AD = AD$ (Common)

$\therefore$ $\triangle ABD \cong \triangle ACD$ (AAS—Congruence)

Then $AB = AC$ (Corresponding parts of congruent triangles.)

Proved

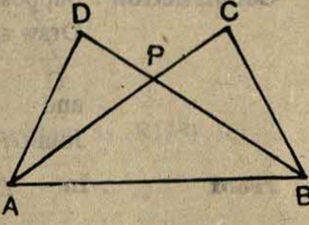
Corollary : An equiangular triangle is also equilateral

EXERCISE—13 (*b*)

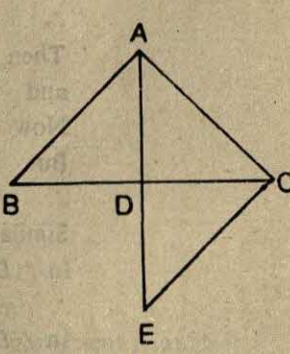
(Section—A)

1. If the base of a triangle is produced both ways and the exterior angles so formed are equal, show that the triangle is isosceles.
2. If from any point on the bisector of an angle, a line is drawn parallel to one arm of the angle, prove that the triangle so formed is isosceles.
3. In an isosceles triangle, a line is drawn parallel to any one side. Prove that the new triangle formed is also isosceles.
4. If the external bisector of an angle of a triangle is parallel to the side opposite to the angle, prove that the triangle is isosceles.

(Section—B)



5. In the figure given alongside, $AD = BC$, $AC = BD$. Prove that PAB is an isosceles triangle.
6. In $\triangle ABC$, the bisectors of $\angle B$ and $\angle C$ meet at P. Through P a line MPN is drawn parallel to BC. Prove that $MN = BM + CN$.



7. If the bisector of the vertical angle of a triangle bisects the base, prove that the triangle is isosceles.

Hint. Produce AD to E so that $DE = AD$. Join EC.

$\triangle ABD \cong \triangle DCE$

$\therefore$ $\angle E = \angle BAD = \angle CAD$

$\therefore$ $AC = CE = AB$

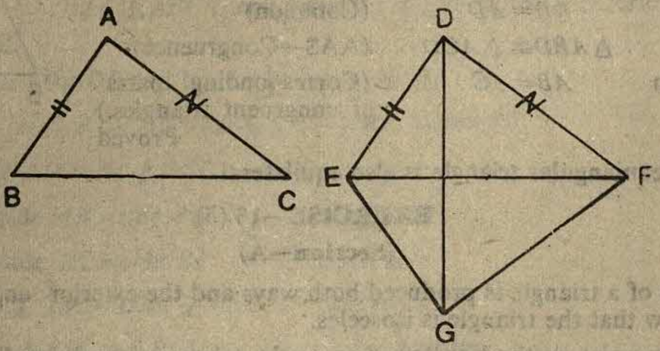
We now give a criterion of congruence involving all the three sides of triangles.

THEOREM 14

(*SSS—Congruence Theorem*)

If three sides of one triangle are congruent to the corresponding sides of the other triangle, the two triangles are congruent.

We refer the above as the *side-side-side* theorem (briefly written as SSS) for congruence of two triangles.



Given : In $\triangle ABC$ and $\triangle DEF$

$AB = DE$

$BC = EF$

and $CA = FD$

To prove : $\triangle ABC \cong \triangle DEF$

Construction : Suppose BC is the *longest* side.

Draw segment EG such that

$EG = AB$

and $\angle FEG = \angle CBA$

Join GF and GD.

Proof : In $\triangle ABC$ and $\triangle GEF$

$AB = GE$ (Construction)

$BC = EF$ (Given)

Incl. $\angle CBA =$ Incl. $\angle FEG$ (Construction)

$\therefore$ $\triangle ABC \cong \triangle GEF$ (SAS—Congruence Axiom)

Then $\angle BAC = \angle EGF$ (Corresponding parts of congruent triangles)

and $CA = FG$

Now $AB = GE$ (Construction)

But $AB = DE$ (Given)

$\therefore$ $DE = GE$

Similarly $FD = FG$

In $\triangle DEG$, $DE = GE$ (Proved above)

$\therefore$ $\angle EGD = \angle EDG$ (Opposite angles to equal sides)

In $\triangle DFG$, $FD = FG$ (Proved above)

$\angle FGD = \angle FDG$ (Opposite angles to equal sides)

$\therefore\ \angle EGD + \angle FGD = \angle EDG + \angle FDG$

i.e., $\angle EGF = \angle EDF$

But $\angle EGF = \angle BAC$ (Proved above)

$\therefore\ \angle BAC = \angle EDF$

Now in $\triangle ABC$ and $\triangle DEF$

$AB = DE$ (Given)

$CA = FD$ (Given)

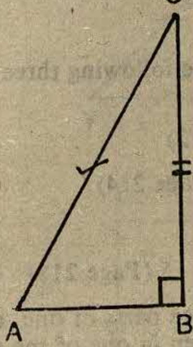
Incl. $\angle BAC =$ Incl. $\angle EDF$ (Proved above)

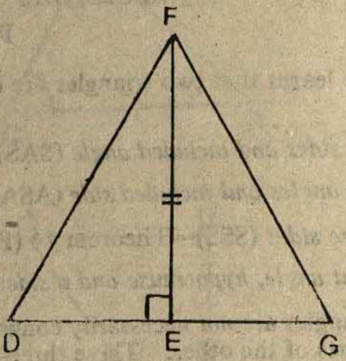
$\therefore\ \triangle ABC \cong \triangle DEF$ (SAS-Congruence Axiom) Proved

THEOREM 15

(*RHS—Congruence Theorem*)

If the hypotenuse and a side of one right triangle are congruent to the corresponding hypotenuse and the side of the other right triangle, the two triangles are congruent.





We refer to the above as *right-hypotenuse-side* theorem (briefly written as RHS) for congruence of two right triangles.

Given : In $\triangle ABC$ and $\triangle DEF$

$\angle B = \angle E = 90°$

$BC = EF$

and hyp. $CA =$ hyp. FD

To prove : $\triangle ABC \cong \triangle DEF$

Construction : Produce DE to G so that $EG = AB$.
Join GF.

Proof : In $\triangle ABC$ and $\triangle GEF$

$AB = GE$ (Construction)

$BC = EF$ (Given)

Incl. $\angle ABC =$ Incl. $\angle GEF$ (Each equal to 90°)

$\therefore\ \triangle ABC \cong \triangle GEF$ (SAS-Congruence Axiom)

Then $\angle A = \angle G$ (Corresponding parts of congruent triangles)

and $CA = FG$

But $CA=FD$ (Given)

$\therefore$ $FD=FG$

In $\triangle DFG$, $FD=FG$ (Proved above)

$\therefore$ $\angle G=\angle D$ (Opposite angles of equal sides)

Then $\angle A=\angle D$ [$\angle A=\angle G$ proved above]

Now in $\triangle ABC$ and $\triangle DEF$

$\angle A=\angle D$ [Proved above]

$\angle B=\angle E$ [Given]

$\therefore$ $\angle C=\angle F$ [Remaining angles of the triangles]

Again in $\triangle ABC$ and $\triangle DEF$

$BC=EF$ [Given]

$CA=FD$ [Given]

and Incl. $\angle C=$ Incl. $\angle F$ [Proved above]

$\therefore$ $\triangle ABC\cong\triangle DEF$ [SAS-Congruence Axiom]

Proved.

We have learnt that two triangles are congruent when the following three parts in each are congruent :

(1) *Two sides and included angle* (SAS)—Axiom (Page 212)

(2) *Two angles and included side* (ASA)—Theorem 12 (Page 214)

(3) *Three sides* (SSS)—Theorem 14 (Page 217)

(4) *Right angle, hypotenuse and a side* (RHS)—Theorem 15 (Page 218)

Two triangles are *not* necessarily congruent when any three parts of one are congruent to any three parts of the other. These three parts must be taken in one of the ways described above.

If the pupils of a class are asked to draw a triangle, having given one of the above sets of conditions, each pupil will draw a triangle.

There will be as many triangles as there are pupils. All the triangles will have the same shape and same size and any two of them will be congruent. They are *not* different triangles but different copies of the *same* triangle.

If the pupils are asked to draw a triangle, having given its three angles, the triangles drawn by the pupils will have the same shape but their sizes will be different. No two of them will be congruent except by chance. Yet all triangles will satisfy the given conditions.

This is so because nothing is said about any side. Even if two angles are given, the third angle is at once known by calculation. This set of conditions in effect reduces to only two angles being given. A third independent part is necessary and its must be a side. Hence we get the following result :

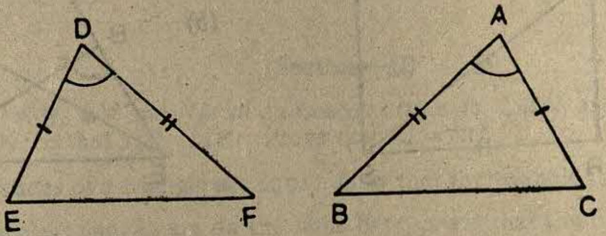
If three angles of a triangle are congruent to the three angles of another, each to each, the triangles may or may not be congruent.

EXERCISE—13 (c)

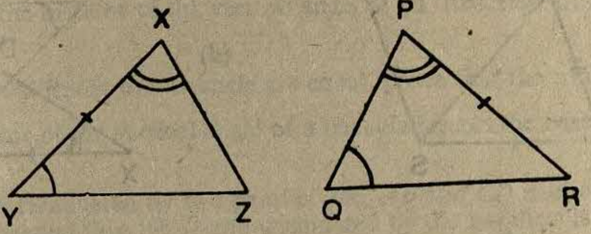
(Section—A)

1. State the reason why the triangles shown in each case are congruent and name the remaining congruent parts.

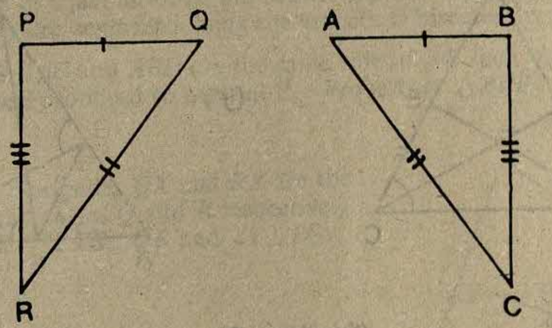
(*a*)



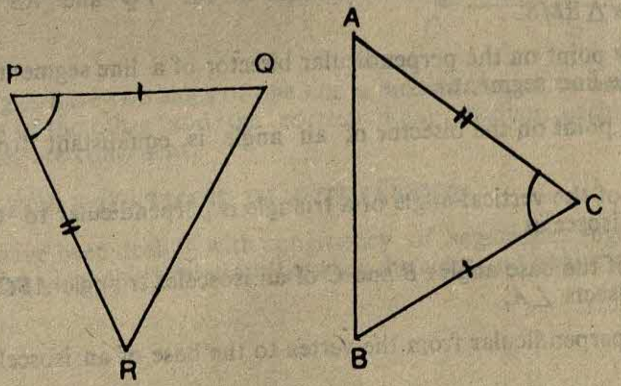
(*b*)



(*c*)

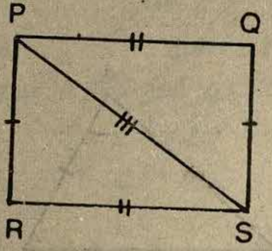


(*d*)

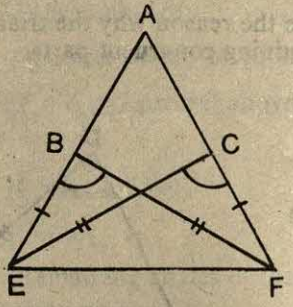


2. Name the congruent triangles in each figure and state why they are congruent.

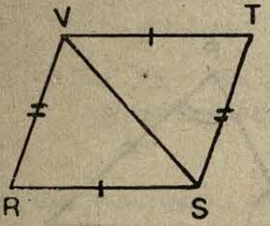
(*a*)



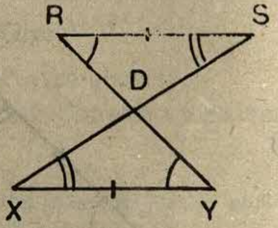
(*b*)



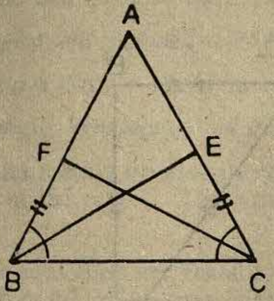
(*c*)



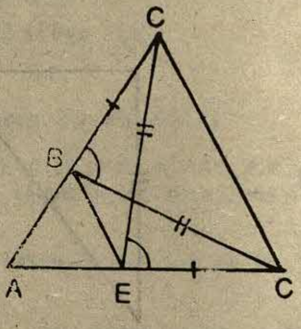
(*d*)



(*e*)



(*f*)



(Section—B)

3. Two segments *PR* and *QS* bisect one another at *M*. *PQ* and *RS* are joined. Prove that $\triangle PMQ \cong \triangle RMS$.

4. Prove that any point on the perpendicular bisector of a line segment is equidistant from the ends of the line segment.

5. Show that any point on the bisector of an angle is equidistant from the arms of the angle.

6. If the bisector of the vertical angle of a triangle is perpendicular to the base, prove that the triangle is isosceles

7. The bisectors of the base angles *B* and *C* of an isosceles triangle *ABC* meet in *O*. Prove that ray *AO* bisects $\angle A$.

8. Show that the perpendicular from the vertex to the base of an isosceles triangle bisects the base.

9. In $\triangle XYZ$, $\angle Y = \angle Z$, prove that the perpendiculars from Y and Z on the opposite sides are equal to one another.

10. (*a*) P is any point in the angle ABC such that the perpendiculars drawn from P on AB and BC are equal. Prove that ray BP is the bisector of angle ABC.

    (*b*) Point P is equidistant from the arms of an $\angle AOB$. Prove that ray OP bisects the angle.

(Section—C)

11. Two equal sides LM and NM of an isosceles triangle LMN are produced to Q and P respectively, so that $MQ = MP$. Prove that $LP = NQ$.

12. If two altitudes of a triangle are equal, show that the triangle is isosceles.

13. If two isosceles triangles are on the same base, prove that the line segment joining their vertices bisects the base at right angles.

14. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

15. If the three altitudes of a triangle are equal, prove that the triangle is equilateral.

16. If the bisector of the vertical angle of a triangle bisects the base, prove that the triangle is isosceles.

17. If a line EF terminated by two parallel lines AB and CD is bisected at O, prove that any other line passing through O and terminated by the parallels is also bisected at O.

18. Two triangles of equal altitudes are drawn on the opposite sides of a common base. Prove that the line segment joining the vertices is bisected by the base.

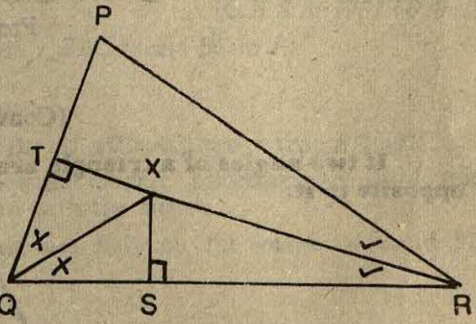
19. Two triangles ABC and ABD are the same side of AB such that $AC = BD$ and $BC = AD$. AC and BD are produced to meet at P. Prove that $\triangle APB$ is isosceles.

20. In the figure alongside, QX and RX are the bisectors of the angles Q and R respectively of the $\triangle PQR$. If $XS \perp QR$ and $XT \perp PQ$.

    Prove that

    (*a*) $\triangle XTQ \cong \triangle XSQ$.

    (*b*) PX bisects the angle P.



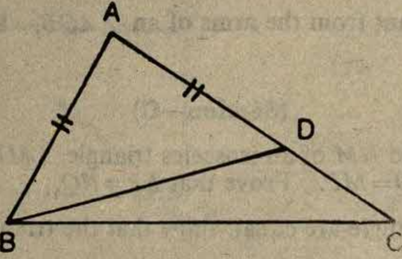
21. If two triangles have two sides of one and a median drawn to one of these sides equal to two sides of the other and the corresponding median, each to each, prove that the two triangles are congruent.

13.3. INEQUALITY RELATIONS IN A TRIANGLE

So far we have been dealing with congruency of segments, angles and triangles. We shall now study some inequality relations between the sides and angles of a triangle.

THEOREM 16

If two sides of a triangle are unequal, the longer side has the greater angle opposite to it.



Given : In $\triangle ABC$, $AC > AB$,

To prove : $\angle ABC > \angle ACB$.

Construction : Mark point D on AC such that $AD = AB$. Join BD.

Proof : In $\triangle ABD$, $AB = AD$ [Construction]

$\therefore$ $\angle ADB = \angle ABD$ [Angles opposite to equal sides]

But $\angle ADB$ is an exterior angle of $\triangle BDC$.

$\therefore$ $\angle ADB > \angle DCB$ [Exterior Angle Theorem]

i.e. $\angle ADB > \angle ACB$

Then $\angle ABD > \angle ACB$ [$\angle ADB = \angle ABD$, proved above]

But $\angle ABD$ is only a part of $\angle ABC$.

$\therefore$ $\angle ABC > \angle ABD$ [Construction]

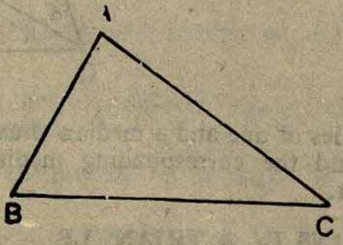
Hence $\angle ABC > \angle ACB$

Proved.

THEOREM 17

(Converse of Theorem 16)

If two angles of a triangle are unequal, the greater angle has the longer side opposite to it.



Given : $\triangle ABC$ in which $\angle ACB > \angle ABC$.

To Prove : $AB > AC$

Proof : When sides AB and AC of $\triangle ABC$ are compared, there are only three possibilities : (*i*) $AB = AC$ (*ii*) $AB < AC$ (*iii*) $AB > AC$

Case (*i*). If $AB = AC$,

then $\angle ACB = \angle ABC$ [Opposite angles to equal sides]

This is contrary to what is given.

$\therefore \quad AB \neq AC$

Case (*ii*). If $AB < AC$ [Longer side has greater angle opposite to it]

then $\angle ABC > \angle ACB$

This is also contrary to what is given.

$\therefore$ AB is not less than AC.

Case (*iii*). Thus AB is neither equal to AC nor less than AC.

So we are left with the only other possibility, namely $AB > AC$.

Proved

EXERCISE—13(d)

(Section—A)

1. In $\triangle ABC$, $AC > AB$ and the bisectors of angles B and C meet at O. Prove that $OC > OB$. [*C.B.S.E. 1979* (*A.I.*)]
2. Prove that in a right-angled triangle the hypotenuse is the greatest side.
3. If the base QR of an isosceles triangle PQR is produced to any point S, show that $PQ < PS$.
4. In $\triangle ABC$, $AB = AC$ and $\angle C = 56°$. Which is the longer side, BC or AC ?
5. Prove that the angle opposite to the greatest side of a triangle is greater than 60°.
6. If the base of an isosceles triangle is greater than each of the equal sides, show that the vertical angle is greater than 60°.

(Section—B)

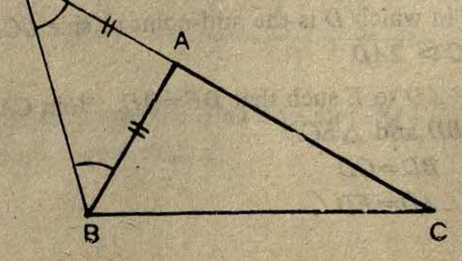
7. AD is the altitude of $\triangle ABC$. Prove that $AB + AC > 2AD$.
8. In $\triangle ABC$, $AB > AC$; the perpendiculars from B and C to the opposite sides intersect at P. Prove that $BP > CP$.
9. In $\triangle ABC$, the bisector of $\angle BAC$ meets BC in D. Prove that $AB > BD$. [*C.B.S E. 1981* (*Delhi*)]
10. In $\triangle ABC$, $\angle B = 40°$, $\angle C = 68°$; the bisector of $\angle BAC$ meets BC in P. Arrange in order of magnitude PA, PB, PC.

(Section—C)

11. In $\triangle ABC$, the bisector of $\angle BAC$ meets BC in D. If $AB > AC$, prove that $BD > DC$.
12. If two sides of a triangle are unequal, prove that the median bisecting the third side makes a greater angle with the smaller side than with the larger.
13. Prove that the bisector of an angle of a triangle lies between the smaller side and the median bisecting the opposite side.

THEOREM 18

The sum of any two sides of a triangle is greater than the third side.



Given : $\triangle ABC$

To Prove : (*i*) $AB+AC>BC$

(*ii*) $BC+AB>AC$

(*iii*) $AC+BC>AB$

Construction : Produce CA to D such that $AD=AB$. Join BD.

Proof : In $\triangle ABD$, $AD=AB$ (Construction)

$\therefore \quad \angle ABD=\angle ADB$ (Opposite angles)

But $\quad \angle CBD>$ its part $\angle ABD$

or $\quad \angle CBD>\angle ADB$ [$\angle ABD=\angle ADB$, proved above]

i.e., $\quad \angle CBD>\angle CDB$

Now in $\triangle BCD$, $\angle CBD>\angle CDB$

$\therefore \quad DC>BC$ [Greater angle has longer side opposite to it].

or $\quad AD+AC>BC$

or $\quad AB+AC>BC$ [$AD=AB$, Construction]

Similarly it can be proved that

(*ii*) $BC+AB>AC$, (*iii*) $AC+BC>AB$

EXERCISE—13(e)

(Section—A)

1. Can a triangle have the sides of the following lengths ?
   (*a*) 3 cm, 4 cm, 5 cm (*b*) 1·5 cm, 3·2 cm, 5·4 cm
   (*c*) 4·5 cm, 7·7 cm, 3·2 cm (*d*) 3·2 cm, 1·6 cm, 2·4 cm
2. Prove shat the semi-perimeter of a triangle is greater than any one of its sides.
3. Prove that any three sides of a quadrilateral are together greater than the fourth side.
4. Prove that the perimeter of a quadrilateral is greater than the sum of its diagonals.

(Section—B)

5. Prove that the difference of any two sides of a triangle is less than the third side.
6. If P is a point in the interior of $\triangle ABC$, prove that
   $$AB+AC > BP+CP$$
   Hint. Produce BP to meet side AC in D.
7. Two sides of a triangle are 6·8 cm and 4·6 cm. Between what limits will the third side lie ?
8. Prove that any two sides of a triangle are together greater than twice the median bisecting the third side.

Solution.

Given : $\triangle ABC$ in which D is the mid-point of side BC.

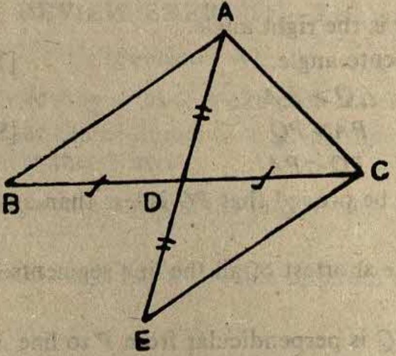
To Prove : $AB+AC > 2AD$

Construction : Produce AD to E such that $DE=AD$. Join CE.

Proof : In $\triangle ABD$ and $\triangle ECD$

$BD=CD$ [Given]

$AD=ED$ [Construction]



Incl. $\angle ADB$ = Incl. $\angle EDC$	[Vert. opp. $\angle s$]
$\therefore \quad \triangle ABD \cong \triangle ECD$	[SAS-Congruence Axiom]
Then $\quad AB = EC$	[Corresponding parts of congruent triangles]
Now in $\quad \triangle ACE$,	
$AC + EC > AE$	[Theorem]
or $\quad AC + AB > AD + DE$	[$EC = AB$, proved above]
or $\quad AB + AC > AD + AD$	[$DE = AD$, Const.]
i.e., $\quad AB + AC > 2AD$	

Proved

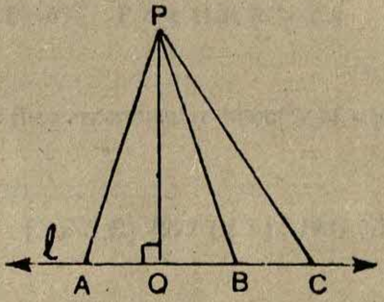
(Section—C)

9. Prove that in any triangle the sum of the medians is less than the sum of the sides.
Hint : Use the result of Exercise No. 8, given above.

10. Prove that the sum of the distances of any point within a triangle from its angular points is less than the perimeter of the triangle.
Hint : Use the result of Exercise No. 6, given above.

THEOREM 19

Of all the line segments that can be drawn to a given line from a given point outside it the perpendicular is the shortest.



Given : l is a line and P is a point not lying on l and $PQ \perp l$.
A is any point on l other than Q.

To Prove : $PQ < PA$.

Proof : In $\triangle PAQ$, $\angle Q$ is the right angle.

$\therefore$ $\angle A$ is an acute angle. [Triangle Angle sum Property]

Then $\angle Q > \angle A$

$\therefore$ $PA > PQ$ [Side opposite to greater angle]

i.e., $PQ < PA$

Similarly it can be proved that PQ is less than any other line segment drawn from P to line l.

Hence PQ is the shortest of all the line segments drawn from P to line l.

Proved

While line segment PQ is perpendicular from P to line l, other line segments PA, PB, PC, etc., are called **oblique** segments.

The distance between a line and a point *not* on it, is the *length of the perpendicular line segment* from the point to the line.

The distance between a line and a point on it is zero.

EXERCISE 13(f)

(Section—A)

1. Prove that the perimeter of a triangle is greater than the sum of its altitudes.
2. Two obliques OP and OQ cut AB at equal distances from C, the foot of the perpendicular ; prove that they are equal.

(Section—B)

3. Obliques OQ and OR are drawn from O to line AB. OC is perpendicular to line AB. If OR cuts line AB at a greater distance from C than OQ, show that $OR > OQ$.
4. Show that it is impossible to draw three equal line segments from a given point to a given line.

REVIEW EXERCISE—XII

(Section—A)

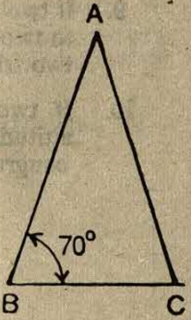
1. Fill in the blanks in the following so as to make each statement true.

(*a*) Of all the segments that can be drawn to a given line from a given point not lying on it, the .... segment is the shortest. [*C.B.S.E., 1977 (A.I.)*]

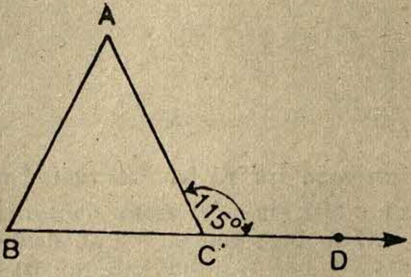
(*b*) If two sides of a triangle are unequal the longer side has the... ...angle opposite to it. [*C.B.S.E., 1978 (Delhi)*]

(*c*) In a triangle, the greater angle has a .....side opposite to it. [*C.B.S.E., 1979 (Delhi)*]

(*d*) In the given figure, $AB=AC$ and $\angle B=70°$. Then $\angle A=$...... [*C.B.S.E., 1979 (A.I.)*]



(*e*) In the given figure, $AB=AC$ and $\angle ACD=115°$. Then $\angle BAC=$........... [*C.B.S.E., 1982 (A.I.)*]



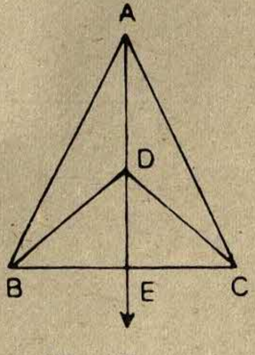
(Section—B)

2. In $\triangle ABC$, $\angle ABC=\angle ACB=65°$. Prove that $BC<CA$. [*C.B.S.E., 1982 (A.I.)*]

3. In the given figure, AD is the perpendicular bisector of segment BC.

Prove that $\angle ABD=\angle ACD$.

[*C.B.S.E., 1977 (A.I.) ; 1981 (Delhi)*]



4. X and Y are two points on equal sides AB and AC of $\triangle ABC$ such that $AX=AY$. Prove that $XC=YB$. [*C.B.S.E., 1983 (Delhi)*]

5. PQR is a triangle in which $PQ=PR$. S and T are points on PQ and PR such that QT and RS are respectively the bisectors of $\angle PQR$ and $\angle QRP$. Prove that
$$\triangle TQR \cong \triangle SRQ.$$
[*C.B.S.E., 1979 (Delhi)*]

6. PQ and RS respectively are the smallest and longest sides of a quadrilateral $PQRS$ Prove that $\angle Q > \angle S$. [*C.B.S.E., 1984 (A.I.)*]

(Section—C)

7. In an equilateral triangle ABC points D, E and F are taken on the sides AB, BC and CA such that $AD=BE=CF$. Prove that DEF is an equilateral triangle.

8. In a right-angled triangle, if one angle is 30°, show that the side opposite to it is half of the hypotenuse.

9. If two triangles have two sides of one and a median drawn to one of these sides equal to two sides of the other and the corresponding median, each to each, prove that the two triangles are congruent.

10. If two sides of one acute-angled triangle are equal to two sides of another and the altitudes from the contained angles are also equal, prove that the triangles are congruent.

□ □

14

PARALLELOGRAMS

14.1. QUADRILATERAL

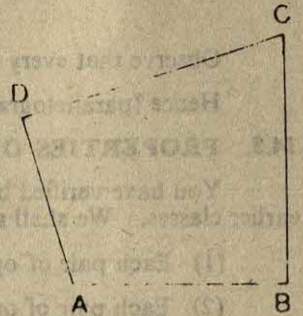
We have already defined a quadrilateral, its sides, vertices, angles and diagonals. We have also proved that the sum of the four angles of a quadrilateral is 360° or 4 right angles. We shall now define some special types of quadrilaterals and discuss some of their important properties.

First we shall learn some terms associated with vertices, sides and angles of a quadrilateral.

Look at the quadrilateral *ABCD* shown on the right.

The vertices of a quadrilateral which are end points of the same side are said to be **adjacent** or **consecutive**. Vertices which are not adjacent or consecutive are called **opposite**. Thus, *A* and *B* are adjacent vertices and *A* and *C* are opposite vertices.

Two sides of a quadrilateral which have a common end point are said to be **adjacent** or **consecutive**. Two sides of a quadrilateral which have no common point are said to be **opposite**.



Thus, sides *AB* and *AD* are adjacent or consecutive and sides *AB* and *DC* are opposite.

Two angles of a quadrilateral which include a side in their intersection are said to be **consecutive** or **adjacent**. Two angles of a quadrilateral which do not include a side in their intersection are said to be **opposite**. Thus, $\angle A$ and $\angle B$ are adjacent and $\angle A$ and $\angle C$ are opposite angles.

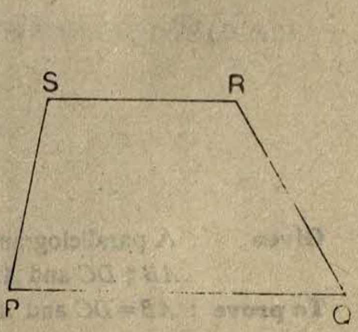
Name the other pair of opposite vertices, sides and angles of quadrilateral *ABCD*.

Also name the other pair of consecutive vertices, sides and angles of quadrilateral *ABCD*.

14.2. TRAPEZIUM AND PARALLELOGRAM

A quadrilateral which has a pair of opposite sides parallel, is called a trapezium.

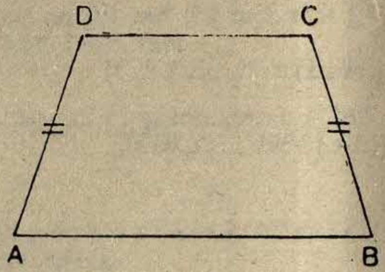
In the quadrilateral *PQRS*, sides *PQ* and *SR* are parallel to each other ; hence *PQRS* is a trapezium.



If the non-parallel sides of a trapezium are equal, it is called an isosceles trapezium.

Here $AB \parallel DC$ and $BC = AD$.

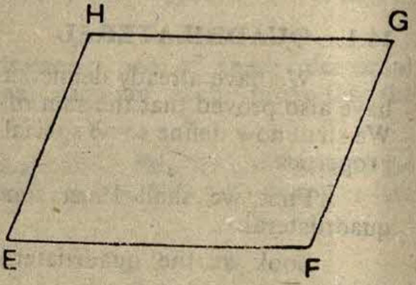
So, *ABCD* is an isosceles tapezium.



A quadrilaterial which has both pairs of opposite sides parallel, is called a parallelogram.

Here $EF \parallel HG$ and $FG \parallel EH$.

So, *EFGH* is a parallelogram.



Observe that every parallelogram is a trapezium and every trapezium is a quadrilateral.

Hence {parallelograms} ⊂ {Trapeziums} ⊂ {Quadrilaterals}.

14.3. PROPERTIES OF A PARALLELOGRAM

You have verified by measurement the following properties of a parallelogram in your earlier classes. We shall now prove them with the help of congruence of triangles.

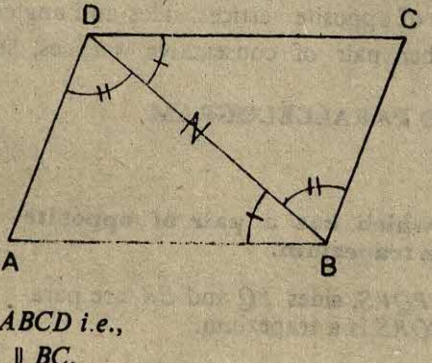
(1) Each pair of opposite sides of a parallelogram are equal.

(2) Each pair of opposite angles of a parallelogram are equal.

(3) The two diagonals of a parallelogram bisect each other.

THEOREM 20 (*a*)

The opposite sides of a parallelogram are equal.



Given : A parallelogram *ABCD i.e.*,
$AB \parallel DC$ and $AD \parallel BC$.

To prove : $AB = DC$ and $AD = BC$.

Const. : Join *BD*.

Proof : In $\triangle ABD$ and $\triangle CDB$

$\angle ABD =$ alternate $\angle CDB$ ($AB \parallel DC$ and DB cuts them.)

$\angle ADB =$ alternate $\angle CBD$ ($AD \parallel BC$ and DB cuts them.)

$BD = BD$ (Common to both)

$\triangle ABD \cong \triangle CDB$ (ASA—congruency)

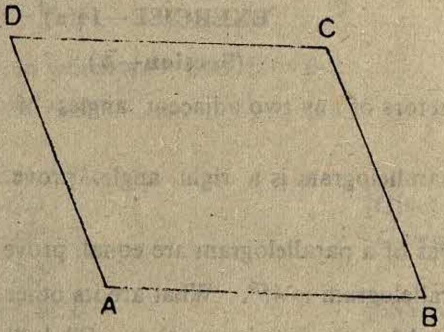
Hence $AB = DC$ and (Corresponding part of congruent triangles.)

$AD = BC$

Proved

THEOREM 20 (*b*)

The opposite angles of a parallelogram are equal.



Given : A parallelogram $ABCD$.

To prove : $\angle A = \angle C$ and $\angle B = \angle D$

Proof : $AB \parallel DC$ and DA cuts them.

$\therefore$ $\angle D + \angle A = 180°$ [Consecutive Interior Angles]

Again $AD \parallel BC$ and DC cuts them.

$\therefore$ $\angle D + \angle C = 180°$ [Consecutive Interior Angles)

Then $\angle D + \angle A = \angle D + \angle C$

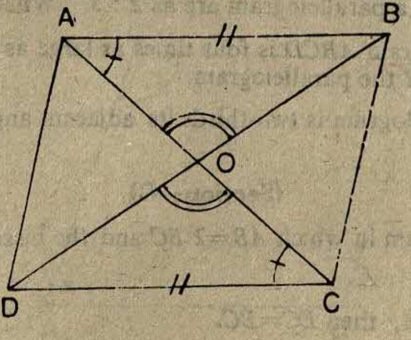
$\therefore$ $\angle A = \angle C$

Similarly, it can be proved that $\angle B = \angle D$ Proved.

THEOREM 20 (*c*)

The diagonals of a parallelogram bisect each other.

[*C.B S E., 1977* (*A.I.*) ; *1978* (*Delhi*)]



Given : A parallelogram $ABCD$ whose diagonals AC and BD intersect at O.

To prove : $AO=OC$ and $BO=OD$.

Proof : In $\triangle AOB$ and $\triangle COD$

$\angle BAO=$alternate $\angle DCO$ [$AB \parallel DC$ and AC cuts them.]

$\angle AOB=$vert. opp. $\angle COD$

and $AB=$corresp. side DC [Opposite sides of a parallelogram]

$\therefore$ $\triangle AOB \cong \triangle COD$ [AAS—congruency]

Hence $AO=OC$ [Corresponding parts of congruent triangles.]

and $BO=OD$

EXERCISE—14(a)

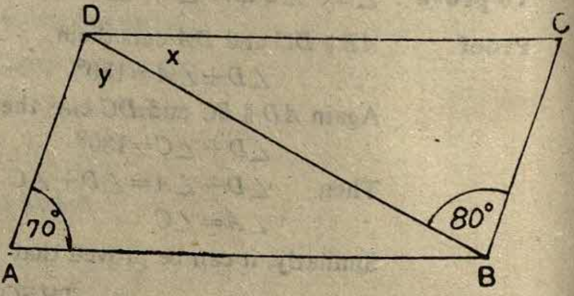
(Section—A)

1. Prove that the bisectors of any two adjacent angles of a parallelogram are at right angles.
2. If one angle of a parallelogram is a right angle, prove that all its angles are right angles.
3. If two adjacent sides of a parallelogram are equal, prove that all its sides are equal.
4. One angle of a parallelogram is 45°. What are its other angles ?
5. One angle of a parallelogram is twice another. Find all the angles.

(Section—B)

6. In the given figure alongside, ABCD is a parallelogram in which $\angle DAB=70°$, $\angle DBC=80°$.

   Calculate angles CDB and ADB.



7. If two opposite angles of a parallelogram contain $(63-3x)°$ and $(4x-7)°$, find the number of degrees in each angle.
8. Two unequal angles of a parallelogram are as 2 : 3. What are all its angles ?
9. Angle A of a parallelogram $ABCD$ is four times as large as angle B. Find the measures of each of the angles of the parallelogram.
10. If an angle of a parallelogram is two-thirds its adjacent angle, find the angles of the parallelogram.

(Section—C)

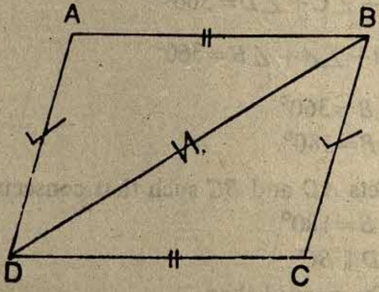
11 $ABCD$ is a parallelogram in which $AB=2\ BC$ and the bisector of $\angle A$ cuts DC in E. Prove that $\angle AEB=1$ rt. $\angle$.

Hint : $DE=DA$, then $EC=BC$.

THEOREM 21 (*a*)

[*Converse of Theorem 20* (*a*)]

A quadrilateral is a parallelogram if its opposite sides are equal.



Given : *A* quad. *ABCD*, in which $AB=CD$, $AD=BC$.

To prove : *ABCD* is a parallelogram.

Const. : Join *BD*.

Proof : In $\triangle ABD$ and $\triangle BCD$

$AB=CD$ [Given]

$AD=BC$ [Given]

$BD=BD$

$\therefore$ $\triangle ABD \cong \triangle BCD$ [SSS-congruency]

$\therefore$ $\angle ABD=\angle BDC$ and $\angle ADB=\angle DBC$ [Corresponding parts of congruent triangles]

Now *BD* intersects *AB* and *DC* such that alternate interior angles.

$\angle ABD=\angle BDC$ [Proved above

$\therefore$ $AB \parallel DC$ ...(1)

Again *DB* intersects *AD* and *BC* such that alternate interior angles,

$\angle ADB=\angle DBC$ [Proved above

$\therefore$ $AD \parallel BC$ (2)

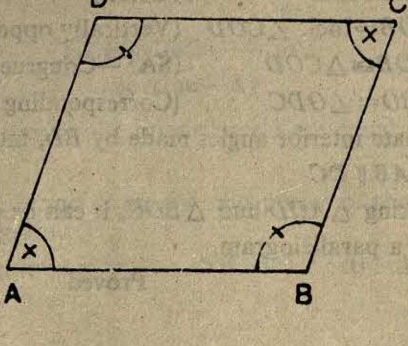
From (1) and (2), *ABCD* is a parallelogram.

Proved

THEOREM 21 (*b*)

[*Converse of Theorem 20* (*b*)]

A quadrilateral is a parallelogram if the opposite angles are equal.



Given : A quad $ABCD$ in which $\angle A = \angle C$ and $\angle B = \angle D$.

To prove : $ABCD$ is a parallelogram.

Proof : In quad. $ABCD$,

$$\angle A + \angle B + \angle C + \angle D = 360^\circ$$

or $\angle A + \angle B + \angle A + \angle B = 360^\circ$ $\left[\begin{array}{l} \angle C = \angle A \\ \angle D = \angle B \end{array}\right.$ given

or $2\angle A + 2\angle B = 360^\circ$

$\therefore$ $\angle A + \angle B = 180^\circ$.

Now AB intersects AD and BC such that consecutive interior angles,

$$\angle A + \angle B = 180^\circ$$

$\therefore$ $AD \parallel BC$

Similarly it can be proved that

$AB \parallel DC$ ...(2)

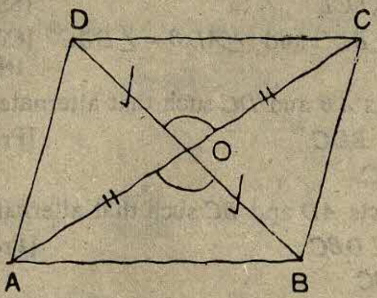
From (1) and (2), $ABCD$ is a parallelogram.

Proved.

THEOREM 21 (*c*)

[*Converse of Theorem 20* (*c*)]

If the diagonals of a quadrilateral bisect each other, it is a parallelogram.



Given : The quad. $ABCD$, with diagonals intersecting at O and having $AO = OC$, $BO = OD$.

To prove : $ABCD$ is a parallelogram.

Proof : In $\triangle AOB$ and $\triangle COD$.

$AO = OC$ (Given)

$BO = OD$ (Given)

Incl. $\angle AOB =$ Incl. $\angle COD$ (Vertically opposite angles)

$\therefore$ $\triangle AOB \cong \triangle COD$ (SAS—Congruency Axiom]

Then $\angle ABO = \angle ODC$ [Corresponding parts of congruent triangles]

These are alternate interior angles made by BD, intersecting AB and DC.

$\therefore$ $AB \parallel DC$

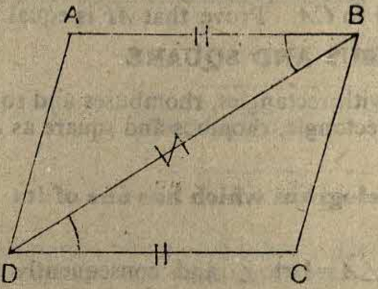
Similarly, by taking $\triangle AOD$ and $\triangle BOC$, it can be proved that $AD \parallel BC$.

Hence $ABCD$ is a parallelogram.

Proved

THEOREM 22

If a pair of opposite sides of a quadrilateral are equal and parallel, it is a parallelogram. [*C.B.S.E., 1977 (A.I.) ; 1981 (A.I.) ; 1982 (Delhi)*]



Given : A quadrilateral $ABCD$ in which $AB \parallel DC$ and also $AB = DC$

To prove : $ABCD$ is a parallelogram.

Const. : Join BD.

Proof : In $\triangle ABD$ and $\triangle CDB$

	$AB = DC$	(Given)
	$BD = BD$	(Common)
Incl	$\angle ABD =$ Incl. $\angle CDB$	[$AB \parallel DC$ and BD cuts them forming alternate interior angles]
$\therefore$	$\triangle ABD \cong \triangle CDB$	(SAS—Congruency Axiom)
Then	$\angle ADB = \angle CBD$	[Corresponding parts of congruent triangles]
$\therefore$	$AD \parallel BC$	[Alternate interior angles being equal]
Thus	$AB \parallel DC$ and	(Given)
	$AD \parallel BC$	(Proved above)
Hence *ABCD* is a parallelogram.		Definition of a parallelogram.
		Proved.

Let us sum up sufficient conditions for a quadrilateral to be a parallelogram.

A quadrilateral is a parallelogram

(1) if both pairs of opposite sides are equal,

or (2) if both pairs of opposite angles are equal,

or (3) if the diagonals bisect each other,

or (4) if a pair of opposite sides are parallel and equal.

EXERCISE—14 (*h*)

(Section—A)

1. If both pairs of opposite angles of a quadrilateral are equal, prove that the quadrilateral is a parallelogram.
2. $ABCD$ is a parallelogram, P and Q are the mid-points of the sides AB and CD : prove that $APCQ$ is a parallelogram.
3. If two lines bisect one another, prove that their extremities form the vertices of a parallelogram.

(Section—B)

4. Points R and S are taken on the diagonal BD of a parallelogram $ABCD$ such that $BR=SD$. Prove that the figure $ARCS$ is a parallelogram.
5. $ABCD$ is a parallelogram. AB is produced to E so that $BE=AB$, EF meets CB produced at F and is parallel to CA. Prove that AF is equal to EC.

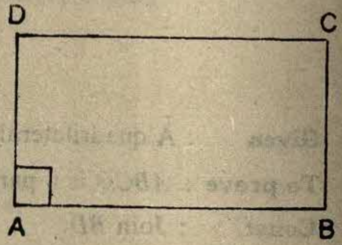
14.3. RECTANGLE, RHOMBUS AND SQUARE

We are already familiar with rectangles, rhombuses and squares. These are particular parallelograms. Let us define rectangle, rhombus and square as a parallelogram with a special property.

A rectangle is a parallelogram which has one of its angles a right angle.

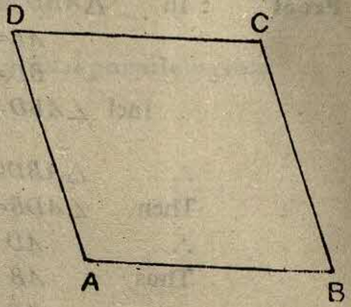
$AB \parallel DC$, $BC \parallel AD$ and $\angle A = 1$ rt. $\angle$ and consequently all angles are right angles.

Thus, a rectangle is a quadrilateral in which all the four angles are right angles.



A rhombus is a parallelogram which has a pair of adjacent sides equal.

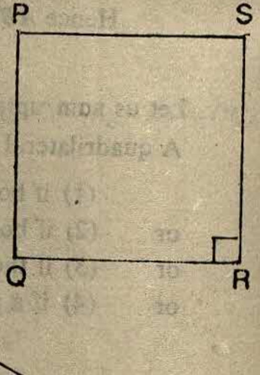
In the parallelogram $ABCD$, $AB=AD$. Therefore all the sides are equal. Thus a rhombus is a quadrilateral in which all the sides are equal.



A square is a rectangle which has a pair of adjacent sides equal.

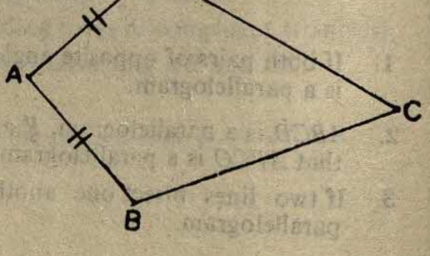
Also a square is a quadrilateral in which all the four sides are equal and all the four angles are right angles.

A rhombus with a right angle is a square.

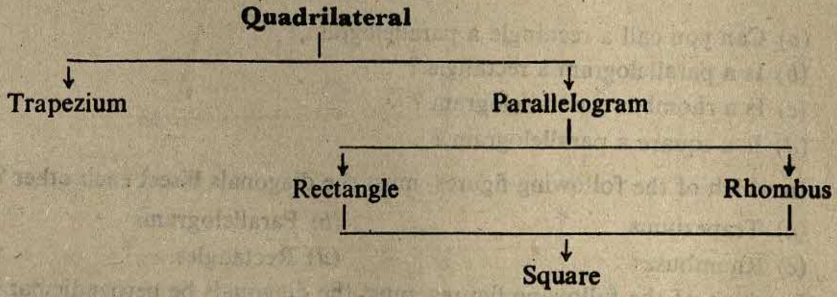


A kite is a quadrilateral which has two pairs of adjacent sides equal.

In the figure $AB=AD$ and $BC=DC$.



Observe the diagram given below, showing different subsets of quadrilaterals and parallelograms.



The diagonals of a parallelogram bisect each other. This remains true for a rectangle, a rhombus or a square because these are also parallelograms.

Let us review the **properties of a parallelogram.**

(*i*) Both pairs of opposite sides are parallel.
Opposite sides are equal.
(*ii*) Opposite angles are equal.
(*iii*) Diagonals bisect each other.
They may or may not be equal.

The following properties of rhombuses, rectangles, and squares follow from the properties of parallelograms :

(*a*) **Properties of Rhombuses**

(*i*) Both pairs of opposite sides are parallel.
All sides are equal.
(*ii*) Opposite angles are equal.
Angles are bisected by the diagonals.
(*iii*) Diagonals bisect each other at right angles.
They may or may not be equal.

(*b*) **Properties of Rectangles**

(*i*) Both pairs of opposite sides are parallel.
Opposite sides are equal.
(*ii*) All angles are equal.
Each angle is a right angle.
(*iii*) Diagonals bisect each other.
Diagonals are equal.

(*c*) **Properties of Squares**

(*i*) Both pairs of opposite sides are parallel.
All sides are equal.
(*ii*) All angles are equal.
Each angle is a right angle.
Angles are bisected by the diagonals.
(*iii*) Diagonals bisect each other at right angles.
Diagonals are equal.

EXERCISE—14 (*c*)

(Section—A)

1. (*a*) Can you call a rectangle a parallelogram ?
   (*b*) Is a parallelogram a rectangle ?
   (*c*) Is a rhombus a parallelogram ?
   (*d*) Is a square a parallelogram ?

2. In which of the following figures, must the diagonals bisect each other ?
   (*a*) Trapeziums (*b*) Parallelograms
   (*c*) Rhombuses (*d*) Rectangles.

3. In which of the following figures, must the diagonals be perpendicular to each other ?
   (*a*) Rectangles (*b*) Rhombuses
   (*c*) Parallelograms (*d*) Squares.

4. Prove that the diagonals of a rectangle are equal.

5. Prove that the diagonals of a rhombus are perpendicular to each other.

6. Prove that the diagonals of a square are equal and perpendicular to each other.

(Section—B)

7. If the two diagonals of a parallelogram are equal, prove that it is a rectangle.

8. If the diagonals of a parallelogram are perpendicular, prove that it is a rhombus.

9. If in a parallelogram, the diagonals are equal and perpendicular, prove that it is a square.

10. Prove that the bisectors of the angles of a parallelogram enclose a rectangle.

11. If the diagonals of a quadrilateral with no angle equal to 90°, bisect each other at right angles, prove that the quadrilateral is a rhombus.

12. If a pair of opposite angles of a parallelogram are bisected by a diagonal, prove that it is a rhombus.

13. If the diagonals of a parallelogram are equal and bisect one other at right angles, prove that it is a square.

(Section—C)

14. *ABCD* is a square. A is joined to a point *P* on *BC* and *D* is joined to a point *Q* on *AB*. If $AP=DQ$, prove that *AP* and *DQ* are perpendicular to each other.

15. A transversal cuts two parallel lines at *A* and *B*. The two interior angles at *A* are bisected and so are the two interior angles at *B* ; the four bisectors form a quadrilateral *ABCD*.

    Prove that (*i*) *ABCD* is a rectangle.
    (*ii*) *CD* is parallel to the original parallel lines.

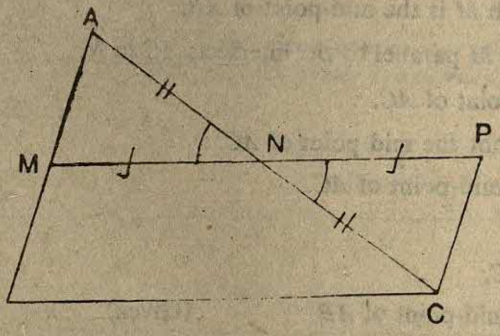
16. *ABCD* is a square. Prove that the straight lines joining vertices *A* and *C* to the mid-points of its opposite sides form a rhombus.

14.4. Using the various theorems on parallelograms, we shall now derive a few useful facts about a triangle.

THEOREM 23

The line segment joining the mid-points of any two sides of a triangle, is parallel to the third side and equal to half of it.

[*C.B.S.E.*, *1979* (*Delhi*) ; *1980* (*A.I.*) ; *1983* (*Delhi*)]



Given : $\triangle ABC$ in which M and N are the middle points of the sides AB and AC respectively.

To prove : $MN \parallel BC$ and $MN=\frac{1}{2}BC$.

Const. : Produce MN to P such that $NP=MN$.
Join CP.

Proof : In $\triangle ANM$ and $\triangle CNP$

$AN=CN$ (N is the mid-point of AC)

$MN=NP$ (Construction)

Incl. $\angle ANM=$ Incl. $\angle CNP$ (Vertically Opposite Angles)

$\therefore$ $\triangle ANM \cong \triangle CNP$ (SAS-Congruency Axiom)

Then $AM=CP$ [Corresponding parts of congruent triangles]

and $\angle MAN=\angle PCN$

But $AM=MB$ (M is the mid-point of AB)

$\therefore$ $MB=CP$ [$AM=CP$, proved above)

Now $\angle MAN=\angle PCN$ (Proved above)

But these are alternate interior angles made by AC intersecting AB and PC.

$\therefore$ $AB \parallel PC$

or $MB \parallel PC$

Thus, in quad. $MPCB$, $MB \parallel PC$ and $MB=PC$

$\therefore$ $MPCB$ is a parallelogram.

Then $MP \parallel BC$ and $MP=BC$ [Opposite sides of a parallelogram]

or $MN \parallel BC$ and $2\,MN=BC$ [$MN=NP$, const.]

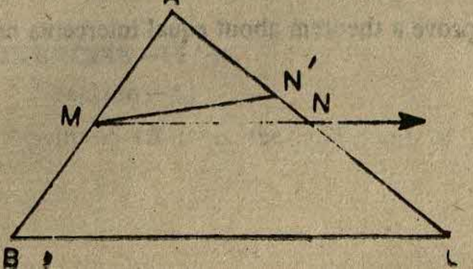
$\therefore$ $MN \parallel BC$ and $MN=\frac{1}{2}BC$.

Proved.

THEOREM 24

(Converse of Theorem 23)

The line drawn through the mid point of one side of a triangle parallel to another side bisects the third side.



Given : $\triangle ABC$ in which M is the mid-point of AB.

A line through M parallel to BC intersects AC in N.

To prove : N is the mid point of AC.

Proof : Suppose N is not the mid-point of AC.

Let N' be the mid-point of AC.

Join MN'.

Now in $\triangle ABC$,

M is the mid-point of AB (Given)

N' is the mid-point of AC (Supposed)

$\therefore$ $MN' \parallel BC$

Also $MN \parallel BC$ (Given)

Thus, two intersecting lines MN and MN' are both parallel to line BC.

This is a contradiction. (Parallel Axiom)

$\therefore$ Our supposition is wrong.

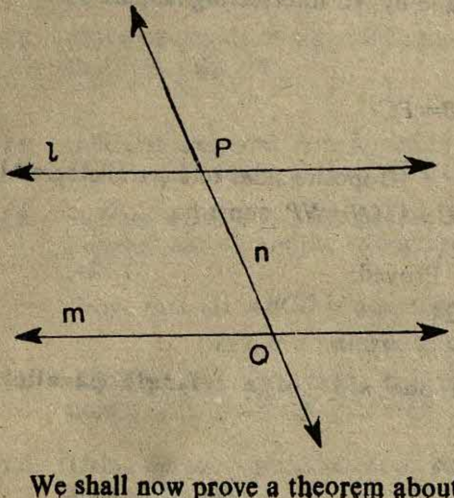
Hence, N is the mid-point of AC.

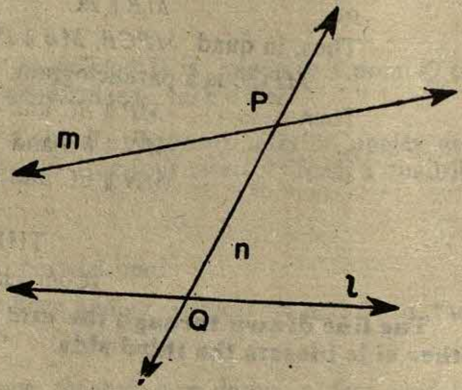
Proved.

14.5. INTERCEPTS

If a transversal n intersects two lines l and m in a plane at P and Q respectively, then line segment PQ is called the **intercept** made on the transversal n by the given lines l and m.

We can also say that lines l and m intercept segment PQ on the transversal n.

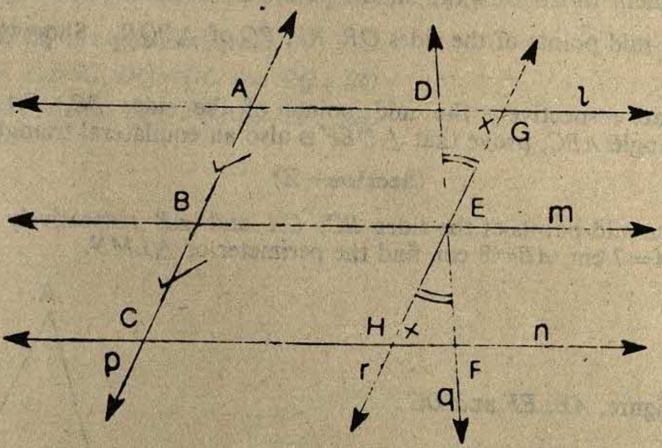




We shall now prove a theorem about equal intercepts made by three parallel lines.

THEOREM 25

If there are three parallel lines and the intercepts made by them on a transversal are equal, then the intercepts on any other transversal are also equal.



Given : Three parallel lines l, m and n.
A transversal p cuts them at A, B and C respectively such that $AB=BC$.
Another transversal q cuts them at D, E and F, respectively.

To prove : $DE=EF$.

Const. : Through E draw line r parallel to transversal p intersecting lines l and n at G and H respectively.

Proof : $AG \parallel BE$ [$l \parallel m$, given]
and $AB \parallel GE$ [$p \parallel r$, construction]
$\therefore$ $AGEB$ is a parallelogram.
Then $AB=GE$ [Opposite sides of the parallelogram]
Similarly, $BEHC$ is a parallelogram. [$BE \parallel CH$ and $BC \parallel EH$]
Then $BC=EH$ [Opposite side of the parallelogram]
But $AB=BC$ [Given]
$\therefore$ $GE=EH$
Now, in $\triangle DEG$ and $\triangle FEH$
$\angle DEG=\angle FEH$ [Vertically opposite angles]
$\angle DGE=\angle FHE$ [Alternate interior angles]
$GE=HE$ [Proved above]
$\therefore$ $\triangle DEG \cong \triangle FEH$ [ASA—Congruency]
Hence $DE=EF$ [Corresponding parts of congruent triangles]

Proved

EXERCISE—14 (d)

(Section—A)

1. In the triangle ABC, D is the mid-point of AB, E is the mid-point of AC. Calculate DE, if $BC=6$ cm.

2. P and Q are the mid-points of the sides AB and AC respectively of $\triangle ABC$. If the length of segment PQ is 3·5 cm, find the length of side BC.

3. Prove that any line segment drawn from the vertex of a triangle to the base is bisected by the line segment which joins the middle points of the other two sides of the triangle.

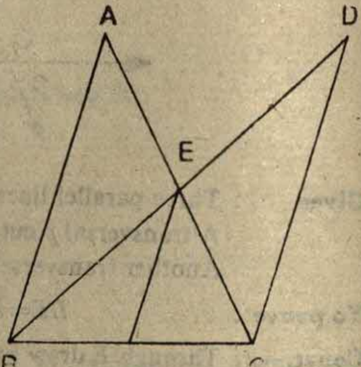
4. X, Y, Z are the mid points of the sides QR, RP, PQ of $\triangle PQR$. Show that $PZXY$ is a parallelogram.

5. If D, E and F are respectively the mid points of the sides BC, CA and AB of an equilateral triangle ABC, prove that $\triangle DEF$ is also an equilateral triangle.

(Section—B)

6. L, M, N are the mid-points of the sides BC, CA and AB respectively of $\triangle ABC$. If $BC=6$ cm, $CA=7$ cm, $AB=8$ cm, find the perimeter of $\triangle LMN$.

7. In the given figure, AB, EF and DC are parallel and $BF=FC$. Prove that $AB=CD$.



8. Prove that the figure formed by joining the mid-points of the pairs of consecutive sides of a quadrilateral is a parallelogram.

9. ABC is a triangle right-angled at B. If D is the mid-point of hypotenuse AC, prove that $BD=\frac{1}{2}AC$. [*C.B.S.E*, *1984* (*A.I.*)]

Hint : Through D, draw a line parallel to BC meeting AB in E.

10. Prove that the line segments joining the middle points of the sides of a triangle divide it into four congruent triangles.

(Section—C)

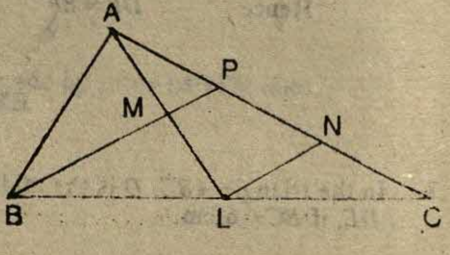
11. Prove that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

Hint : Use Exercise No. 8, given above.

12. ABC is an isosceles triangle in which $AB=AC$; D, E and F are respectively the mid-points of sides BC, AB and CA. Prove that segments AD and EF bisect each other at right angles.

13. In the figure, L is the mid-point of side BC of $\triangle ABC$; AL is bisected at M and BM produced cuts AC in P. Prove that $AP=\frac{1}{3}AC$.

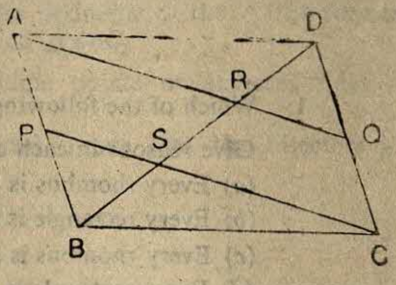
Hint : Bisect PC at N. Join LN.



14. In the figure, *P* and *Q* are respectively the mid-points of the sides *AB* and *CD* of a parallelogram *ABCD*.

Prove that *DR*=*RS*=*SB*. [*C.B.S.E., 1983 (A.I.)*]

Hint : *APCQ* is a parallelogram.
In $\triangle DSC$, *DQ*=*QC* and *RQ* ∥ *SC*
In $\triangle ARB$, *AP*=*PB* and *PS* ∥ *AR*.



REVIEW EXERCISE—XIII

(Section—A)

1. Which of the following statements are True and which are False ?

Give reasons in each case.

(*a*) Every rhombus is a parallelogram.
(*b*) Every rectangle is a rhombus.
(*c*) Every rhombus is a rectangle.
(*d*) Every rectangle is a parallelogram.
(*e*) Every square is a rectangle.
(*f*) Every rhombus is a square.
(*g*) Every square is a rhombus.
(*h*) Every square is a parallelogram.
(*i*) Every rectangle is a square.
(*j*) Every trapezium is a parallelogram.

2. Fill in the blanks to make the following statements true :

(*a*) The opposite angles of a parallelogram are ……… . [*C.B.S.E. 1978 (A.I.)*]

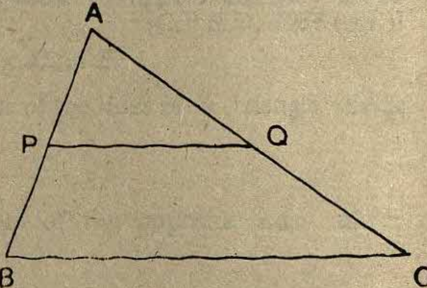
(*b*) A parallelogram is a rhombus if and only if its diagonals are ……… . [*C.B.S.E., 1979 (Delhi)*]

(*c*) A quadrilateral is a parallelogram if and only if its diagonals ……… each other. [*C.B.S.E., 1980 (A.I.)*]

(*d*) Diagonals of a rhombus are ……… to each other. [*C.B.S.E., 1981 (A.I.)*]

(*e*) The line drawn through the mid-point of one side of a triangle is parallel to another side ……… the third side. [*C.B.S.E., 1977 (Delhi)*]

(*f*) In the figure, if P and Q are respectively the mid-points of the sides AB and AC of $\triangle ABC$, then $PQ=$ ‥ cm. [*C.B.S.E., 1982 (Delhi)*]



(*g*) If the length of the segment joining the mid-points of sides AB and AC of $\triangle ABC$ is 7 cm, then length of side BC is ……… . [*C.B.S.E., 1977 (A.I.)*]

3. What two facts may be stated about the diagonals of a rectangle ?

4. What three facts may be stated about the diagonals of a rhombus ?

5. What four facts may be stated about the diagonals of a square ?

(Section—B)

6. $ABCD$ and $ABEF$ are two parallelograms on opposite sides of AB. Prove that $DCEF$ is also a parallelogram.

7. $ABCD$ is a parallelogram in which X and Y are points on the diagonal BD such that $DX=BY$. Prove that $AXCY$ is a parallelogram.

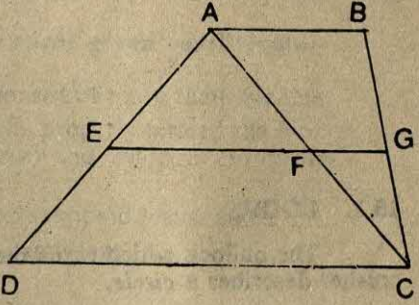
8. ABC is a triangle. Through vertices A, B and C three lines are drawn respectively parallel to their opposite sides. Prove that the perimeter of the $\triangle DEF$ formed by these three lines is double the perimeter of the given $\triangle ABC$.

9. Prove that the figure formed by joining the middle points of adjacent sides of a rhombus is a rectangle.

10. If two parallel lines are intersected by a transversal, prove that the bisectors of the interior angles form a rectangle.

(Section – C)

11. In the given figure, $ABCD$ is a trapezium in which side AB is parallel to side DC and E is the mid-point of side AD. If G is a point on BC such that segment $EG \parallel$ side DC, then prove that $EG=\frac{1}{2}(AB+DC)$.
[*C.B.S.E., 1982 (Delhi)*]



Hint : Join AC cutting EG at F.
In $\triangle ACD$, $AF=FC$ and $EF=\frac{1}{2}DC$.
In $\triangle ABC$, G is the mid-point of BC.

12. Prove that the line segment joining the mid-points of the non-parallel sides of a trapezium is parallel to each of the parallel sides and is equal to half of the sum of the lengths of the parallel sides.

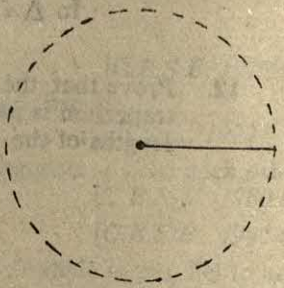
□ □

15

LOCI AND CONCURRENCY

15.1. LOCUS

The bullock which turns the oilman's press or the sugarcane crusher describes a *circle.*

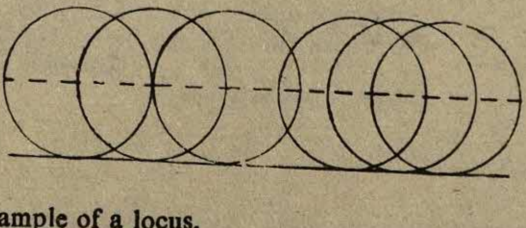


Why does the bullock make a circle track ?

It just cannot do otherwise, Its movement is restricted by the frame to which it is yoked. The frame keeps it at a fixed distance from the central machine.

The motion of the bullock can be explained in geometrical terms. A new term used for this purpose is "**Locus**". It means a *path,* or a *curve* or a *track.*

Take a single cycle wheel, fix a brush carrying ink to the axle at its centre and roll the wheel along side a wall. The brush leaves a line mark on the wall at a height equal to the radius of the wheel.

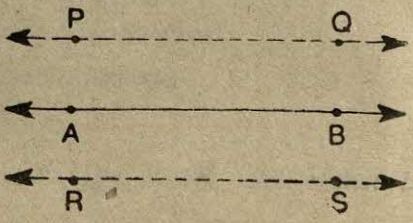


This is another example of a locus.

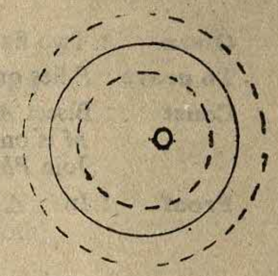
Let us consider two more examples.

Suppose a point moves so that it always keeps itself at a distance of 5 cm from a given line *AB.*

If we join the various positions that it can take up the *path* will be a *pair of parallel* lines *PQ* and *RS* on, either side of *AB* at a distance of 5 cm from *AB.*



A circle with centre O is given. A point moves so that it always keeps itself at a distance of 3 cm from the circumference. The point can move either *inside* or *outside* the given circle. So *each of the* two *dotted circles* represents the *path* traced out by the point which always keeps itself at a distance of 3 cm from the circumference.



Observe that in each case, the points satisfy a certain condition. And in each case, we have a locus of points. Thus, *the path traced out by a point which moves subject to certain given geometrical conditions is* the locus of the point.

A locus of points is the set of all points which satisfy some given condition(s).

Note : Plural of locus is *loci* (pronounce losai).

The locus of a point need not always be a circle or it may consist of more than one line.

In order to prove that a path (a line or set of lines) may be properly termed the *locus* of a point which satisfies a given geometrical condition, it is *necessary* and *sufficient* to prove two facts :

(*i*) that every point satisfying the given condition lies on the supposed locus, and

(*ii*) that any point on the supposed locus satisfies the given condition.

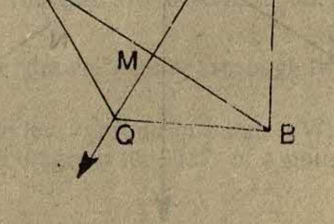
EXERCISE—15 (a)

(Section—A)

1. A stone is tied to one end of a string. The other end is held in the hand and it is whirled. What is the locus of the stone ?
2. What is the locus of all points 3·5 cm away from a given line ?
3. What is the locus of all the students who stand at a distance of 10 metres from the school flag pole ?
4. A point P moves so that its distance from a fixed line AB is always the same. What is the relation between AB and the path travelled by P ?
5. P is a fixed point and a point Q moves such that the distance PQ is constant. What is the locus of the path traced out by the point Q ?
6. You are asked to place a coin so that it is equidistant from (*a*) two adjacent edges, (*b*) two opposite edges of a carromboard. Where will you place it ?
7. A point P is allowed to travel in space. State the locus of P so that it always remains at a constant distance from a fixed point C.

THEOREM 26

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the line segment joining the two fixed points.



PART—I

Given : Two fixed points A and B and a point P which moves such that $PA=PB$.

To prove : P lies on the right bisector of AB.

Const : Bisect AB at M.
M is one position of P.
Join PM.

Proof : In $\triangle PAM$ and $\triangle PBM$

$PA=PB$ (Given)
$AM=BM$ (Construction)
and $PM=PM$ (Common to both,)
$\therefore$ $\triangle PAM\cong\triangle PBM$ SSS—Congruency-Theorem
Then $\angle PMA=\angle PMB$ (Corresponding parts of congruent triangles)
But $\angle PMA+\angle PMB=2$ rt. $\angle$s (Linear Pair Axiom)
$\therefore$ $\angle PMA=\angle PMB=1$ rt. $\angle$ (Proved above)
i.e. $PM\perp AB$
PM is the right-bisector of AB. [M is the mid-point of AB]
i.e., P lies on the right-bisector of AB.

PART—II

Conversely : Let Q be any point on PM, the right-bisector of AB. Join QA, QB.

To prove : $QA=QB$

Proof : In $\triangle QAM$ and $\triangle QBM$

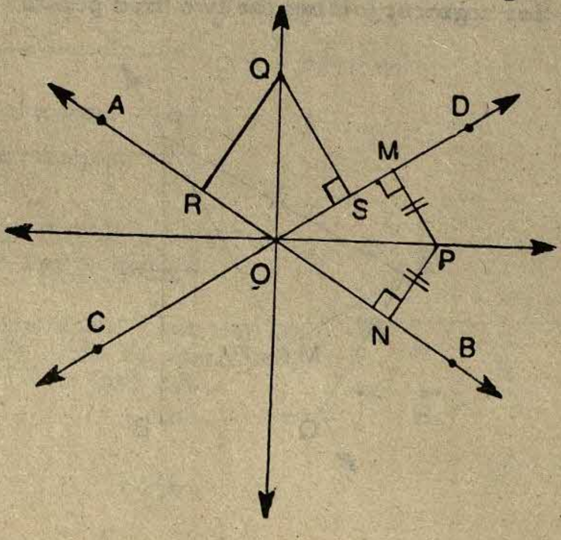
$AM=BM$ (Construction)
$QM=QM$ (Common to both)
and incl. $\angle AMQ=$incl. $\angle BMQ$ (PQ bisects AB at M at rt. $\angle$s.)
$\therefore$ $\triangle QAM\cong\triangle QBM$ (SAS—Congruency Axiom)
Then $QA=QB$ [Corresponding parts of congruent triangles]

Hence, any point Q on the right-bisector of AB is equidistant from A and B.

Thus, the locus of a point equidistant from two fixed points A and B is the right-bisector of the line segment joining A and B

THEOREM 27

The locus of a point which is equidistant from two intersecting lines consists of a pair of lines which bisects the angles between the two given lines.



PART—I

Given : Two lines AB and CD intersecting at O.
A point P within $\angle BOD$ such that perp. PM=perp. PN.
OP is joined.

To prove : P lies on the bisector of the $\angle BOD$.

Proof : In the rt. $\angle$d $\triangle POM$ and $\triangle PON$

hyp. OP=hyp. OP (Common to both)
side PM=sine PN (Given)
$\triangle POM \cong \triangle PON$ [RHS—Congruency Theorem]
Then $\angle POM = \angle PON$ [Corresponding parts of congruent triangles
i.e., OP bisects $\angle BOD$.
or P lies on the bisector of $\angle BOD$

PART—II

Conversely : Let Q be a point on the bisector OQ of $\angle AOD$ and QR and QS be $\perp$ AB and CD respectively.

To prove : $QR=QS$

Proof : In the $\triangle QRO$ and $\triangle QSO$

rt. $\angle QRO$=rt. $\angle QSO$ (Given)
$\angle QOR = \angle QOS$ (Given)
and $QO=QO$ (Common side)
$\therefore$ $\triangle QRO \cong \triangle QSO$ [ASA—Congruency Theorem]
Then $QR=QS$ [Corresponding parts of congruent triangles]
i.e., Q is equidistant from AB and CD.

Thus, any point on the bisector of the angle between AB and CD is equidistant from AB and CD.

EXERCISE—15 (b)

(Section—A)

1. State the locus of a point, which is equidistant from two end points of a fixed line segment.
2. A and B are two fixed points and a point P moves such that it is equidistant from A and B. What is the locus of the path traced out by the point P.
3. A point P moves so that its perpendicular distance from two given lines AB and CD are equal. State the locus of the point P.
4. AB is a fixed line segment ; state the locus of the point P so that $\angle APB=90°$.
5. State the locus of a point in rhombus $ABCD$ which is equidistant from
   (*i*) AB and AD, and
   (*ii*) A and C.
6. Several isosceles triangles are described on a given base. What is the locus of their vertices ?
7. Several circles are drawn passing through two fixed points. Where do their centres lie ?
8. A wheel is rolled on level ground in a line. What is the locus of its centre ? Can you indicate by a rough diagram the locus of a point on the edge of the above wheel ?

(Section—B)

9. From a given point several lines are drawn to meet another line. What is the locus of the middle points of all these lines ?

10. On the same base on the same side of it several triangles of equal area are described. What is the locus of their vertices ?

11. A man steps on the middle rung of a ladder which leans against a wall ; the ladder slides down. Find the locus of his foot.

12. Fix two pins A, B on a drawing board. Place a set square between them so that its two sides touch the pins. Rotate the set-square, taking care to see that the edges always touch the pins. What is the locus of its right-angled corner ?

13. $\angle XOY$ is 45°. A point P moves so that its distance from OX is greater than that from OY by 1 cm. Find the locus of P.

14. $\angle XOY$ is 60°. A point P moves so that the sum of its distances from OX and OY is 5 cm. Find its locus.

(Section—C)

15. $\triangle PBC$, $\triangle QBC$ and $\triangle RBC$ are three isosceles triangles on the same base BC. Prove that P, Q and R are collinear.

Hint : P lies on the perpendicular bisector of line segment BC.
Q lies on the perp bisector of segment BC.
R lies on the perp bisector of segment BC.

16. The bisectors of $\angle B$ and $\angle C$ of a quadrilateral $ABCD$ intersect in P. Prove that P is equidistant from the opposite sides AB and CD

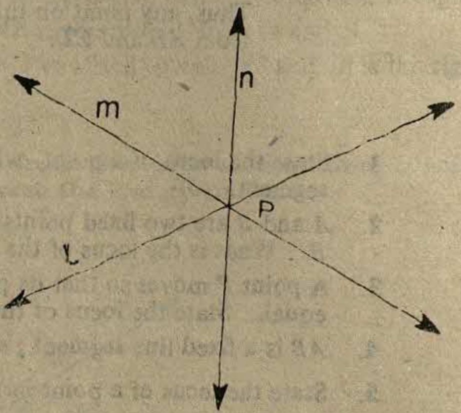
15.2. CONCURRENT LINES

Three or more lines are said to be concurrent lines if they all pass through the same point.

The common point or the point of intersection is called the **point of concurrency**.

Here lines l, m and n pass through the point P, so l, m and n are concurrent lines. Point P is the point of concurrency of lines l, m and n.

Observe that point P lies on each of the given lines.



Draw a triangle and draw its three medians.

Are these medians concurrent ?

Draw a triangle and also draw the bisectors of its angles.

Are these angle bisectors concurrent ?

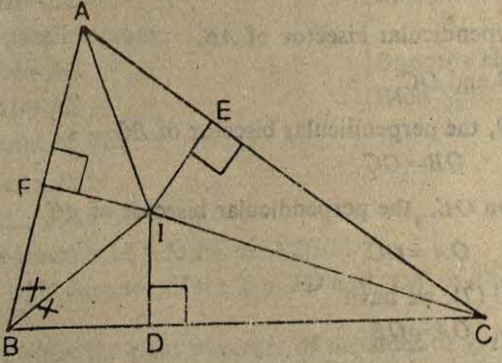
Draw a triangle and also draw its altitudes.

Are these altitudes of the triangle concurrent ?

You have already learnt about concurrency of (*i*) the angle bisectors, (*ii*) the perpendicular bisectors of the sides, (*iii*) the altitudes, and (*iv*) the medians of the sides of a triangle. We shall now state and prove four theorems about these cases of concurrency.

THEOREM 28

The angle bisectors of a triangle are concurrent.



Given : $\triangle ABC$ in which BI and CI, the bisectors of $\angle B$ and $\angle C$ respectively intersect at I. AI is joined.

To prove : AI bisects $\angle A$.

Const. : Draw $ID \perp BC$, $IE \perp CA$ and $IF \perp AB$.

Proof : I lies on the bisector of $\angle B$.

$$\therefore \quad ID = IF \qquad \ldots(1)$$

Also I lies on the bisector of $\angle C$.

$$\therefore \quad ID = IE \qquad \ldots(2)$$

From (1) and (2), we have

$$IE = IF$$

i.e., I is equidistant from AB and AC.

$\therefore$ I lies on the bisector of $\angle A$.

i.e., AI bisects $\angle A$.

Proved

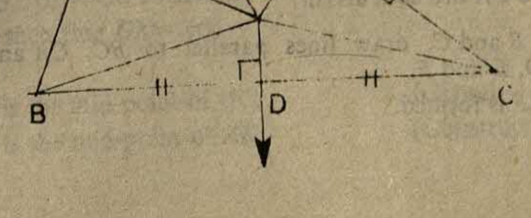
The point where the bisectors of the angles of a triangle meet is called the incentre of the triangle.

To determine the incentre of a triangle, it is just sufficient to find the point of intersection of its two angle bisectors. The third angle bisector will automatically pass through the same point, by virtue of the above theorem.

THEOREM 29

The perpendicular bisectors of the sides of a triangle are concurrent.

[*C B.S.E., 1980 (वं I.)*]



Given : $\triangle ABC$ in which DO and EO, the perpendicular bisectors of BC and CA respectively, intersect at O.

Also, $OF \perp AB$ is drawn.

To prove : OF is the perpendicular bisector of AB.

Const. : Join OA, OB and OC.

Proof : O lies on OD, the perpendicular bisector of BC.

$\therefore \quad OB = OC$ ...(1)

Also O lies on OE, the perpendicular bisector of AC.

$\therefore \quad OA = OC$ ...(2)

From (1) and (2), we have

$OA = OB$

i.e., O is equidistant from A and B.

$\therefore$ O lies on the perpendicular bisector of AB.

Then OF is the perpendicular bisector. [$OF \perp AB$]

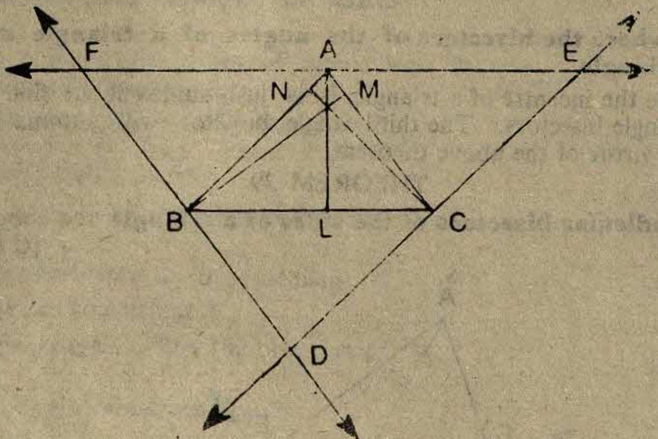
Proved

The point where the right-bisectors of the sides of a triangle meet is called the circumcentre of the triangle.

To determine the circumcentre of a triangle, it is just sufficient to find the point of intersection of the perpendicular bisectors of its two sides. The perpendicular bisector of the third side will automatically pass through the same point. by virtue of the above theorem.

THEOREM 30

The altitudes of a triangle are concurrent.



Given : $\triangle ABC$ in which AL, BM and CN are the altitudes.

To prove : AL, BM and CN are concurrent.

Const. : Through A, B and C. draw lines parallel to BC, CA and AB respectively intersecting at D, E and F.

Thus $\triangle DEF$ is formed.

Proof : $AE \parallel BC$ and $CE \parallel BA$. [Construction]

$\therefore$ $ABEC$ is a parallelogram.

Then $AE=BC$ [Opposite sides]

Similarly $FBCA$ is a parallelogram.

Then $FA=BC$ [Opposite sides]

$\therefore$ $FA=AE$ [Each equal to BC]

Then A is the mid-point of FE.

Now, $FE \parallel BC$ and $AL \perp BC$.

$\therefore$ $AL \perp FE$.

Thus, AL is the right-bisector of side FE of $\triangle DEF$.

Similarly, BM is the right-bisector of the side FD and CN is the right bisector of the side DE of $\triangle DEF$.

Thus, AL, BM and CN are the right bisectors of the sides of $\triangle DEF$.

$\therefore$ AL, BM and CN are concurrent.

i.e., the altitudes of a triangle are concurrent.

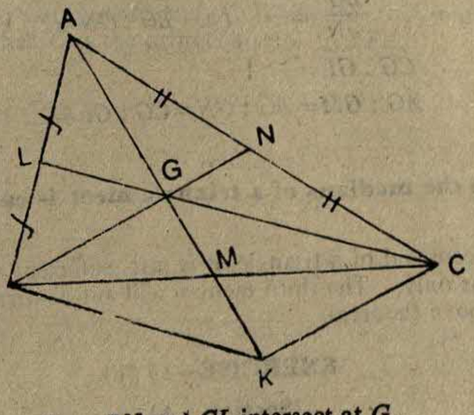
Proved

The point where the altitudes of a triangle meet is called the orthocentre of the triangle.

To determine the orthocentre of a triangle, it is just sufficient to find the point of intersection of its two altitudes only. The third altitude will automatically pass through the same point, by virtue of the above theorem.

THEOREM 31

The medians of a triangle pass through the same point which divides each of the medians in the ratio 2 : 1. [*C.B.S.E., 1982 (A.I.)* ; *1984 (Delhi)*]



Given : $\triangle ABC$ in which medians BN and CL intersect at G.
AG is joined and produced to meet BC in M.

To Prove : (*i*) $BM=MC$
(*ii*) $AG : GM=BG : GN=CG : GL=2 : 1$

Const. : Produce AG to K such that $GK=AG$.
Join BK and CK.

Proof : (*i*) In $\triangle ACK$, N is the mid-point of AC. [Given]
G is the mid-point of AK. (Construction)

$\therefore$ $GN \parallel KC$ and $GN=\frac{1}{2}\,KC$

or $BG \parallel KC$ and $GN=\frac{1}{2}\,KC$ ...(1)

In $\triangle ABK$, L is the mid-point of AB (Given)

G is the mid-point of AK (Construction)

$\therefore$ $LG \parallel BK$ and $LG=\frac{1}{2}\,BK$.

or $GC \parallel BK$ and $LG=\frac{1}{2}\,BK$ ...(2)

From (1) and (2), we have

$BKCG$ is a parallelogram.

$\therefore$ Diagonals BC and GK of parallelogram $BKCG$ bisect each other.

Then $BM=MC$ and $GM=MK$.

i.e., M is the mid-point of BC.

or AGM is the third median.

i.e., the three medians of a triangle are concurrent.

(*ii*) Now $CM=MK$ (Proved above)

or $GK=2\,GM$

But $GK=AG$ (Construction)

$\therefore$ $AG=2\,GM$

or $\dfrac{AG}{GM}=\dfrac{2}{1}$ *i.e.*, $AG:GM=2:1$

Again, $GN=\frac{1}{2}\,KC$ [Proved above (1)]

$=\frac{1}{2}\,BG$ [$KC=BG$, opposite side of parallelogram $BKCG$]

or $\dfrac{BG}{GN}=\dfrac{2}{1}$ *i.e.* $BG:GN=2:1$

Similarly $CG:GL=2:1$

Hence, $AG:GM=BG:GN=CG:GL=2:1$

Proved

The point where the medians of a triangle meet is called the centroid of the triangle.

To determine the centroid of a triangle, it is just sufficient to find the point of intersection of its two medians only. The third median will automatically pass through the same point, by virtue of the above theorem.

EXERCISE—15 (c)

(Section—A)

1. (*a*) Can the centroid lie outside a triangle ?

   (*b*) Can the centroid lie on any side of a triangle ?

2. What is the position of the circumcentre in the following cases ?

   (*a*) an acute-angled triangle

   (*b*) a right-angled triangle

   (*c*) an obtuse-angled triangle.

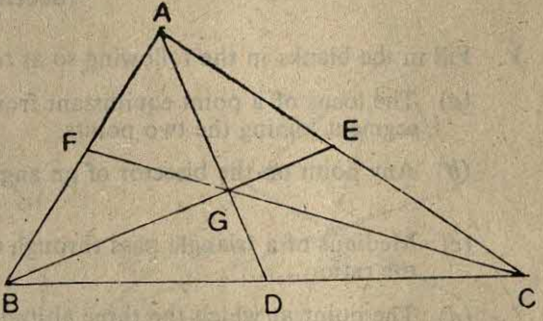
3. Does the orthocentre of a triangle always lie in its interior ?

4. In $\triangle ABC$, the medians AD, BE and CF pass through G.

   (a) If $AG = 2{\cdot}4$ cm, what is GD ?

   (b) If $BE = 4{\cdot}5$ cm, find BG.

   (c) If $GF = 1{\cdot}9$ cm, what is CF ?



(Section—B)

5. Can the incentre of a triangle coincide with its circumcentre and orthocentre ? In which kind of triangle ?

6. Prove that the circumcentre of a right-angled triangle coincides with the mid-point of its hypotenuse.

7. If O is the orthocentre of $\triangle ABC$, prove that B is the orthocentre of $\triangle AOC$ and A is the orthocentre of $\triangle BOC$.

8. If two medians of a triangle are equal, prove that the triangle is isosceles.

(Section—C)

9. If the medians of a triangle are equal to each other, prove that it is equilateral.

10. Prove the internal bisector of $\angle A$ and the external bisectors of $\angle B$ and $\angle C$ of $\triangle ABC$ are concurrent.

11. If O is the orthocentre of $\triangle ABC$, and X, Y, Z are the mid-points of AO, BO, CO respectively, prove that O is the orthocentre of $\triangle XYZ$.

REVIEW EXERCISE—XIV

(Section—A)

1. Fill in the blanks in the following so as to make each of the statements true :

   (*a*) The locus of a point equidistant from two given points is the.........bisector of the segment joining the two points. [*C.B.S.E.*, *1977* (*Delhi*)]

   (*b*) Any point on the bisector of an angle is .... from the arms of the angle. [*C.B.S.E.*, *1978* (*A.I.*)]

   (*c*) Medians of a triangle pass through the same point which divides each of them in the ratio......... . [*C B.S.E.*, *1980* (*Delhi*)]

   (*d*) The point at which the three altitudes of a triangle meet is called the........ of the triangle. [*C.B.S.E.*, *1984* (*A.I.*)]

   (*e*) The perpendicular bisectors of the sides of a triangle pass through the...........point.

   (*f*) The point of concurrence of the medians of a triangle is called............of the triangle.

   (*g*) The bisectors of the angles of a triangle are........... .

2. What is the position of the orthocentre in the following cases :

   (*a*) an acute-angled triangle

   (*b*) a right-angled triangle

   (*c*) an obtuse-angled triangle.

3. (*a*) Is the circumcentre of a triangle equidistant from its vertices ?

   (*b*) Is the incentre of a triangle equidistant from its sides ?

   (*c*) Is the centroid of a triangle equidistant from its sides ?

   (*d*) Is the orthocentre of a triangle equidistant from its vertices ?

(Section—B)

4. In $\triangle ABC$, the medians AD, BE and CF intersect in G.

   Prove that $BE+CF>\frac{3}{2}BC$.

   Hint : In $\triangle BGC$, $BG+CG>BC$.

5. If the line segment joining the incentre and circumcentre of a triangle is perpendicular to a side, prove that the triangle is isosceles.

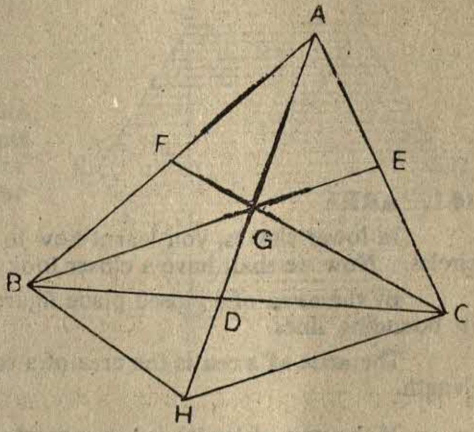
6. $\triangle ABC$ is isosceles with $AB=AC$. D is the mid-point of BC. Show the circumcentre, the incentre, the orthocentre and the centroid all lie on line AD.

7. $ABCD$ is a parallelogram. E and F are the circumcentres of $\triangle ABD$ and $\triangle BCD$ respectively. Prove that $BEDF$ is a rhombus.

8. Q is any point on the base of an isosceles $\triangle ABC$. P and R are the circumcentres of $\triangle ABQ$ and $\triangle ACQ$ respectively. Prove that $APQR$ is a rhombus.

9. In $\triangle ABC$, medians AD, BE and CF intersect in G. Prove that
$4(AD+BE+CF)>3(AB+BC+CA)$

Hint : Use Exercise No. 4, given above.



10. Prove that the sum of any two medians of a triangle is greater than the third.

Hint : In $\triangle GCH$, $GC+CH>GH$

or $GC+GB>AG$

or $\frac{2}{3}CF+\frac{2}{3}BE>\frac{2}{3}AD$.

(Section—C)

11. The vertex B of parallelogram $ABCD$ is joined to X and Y, the mid-points of AD and DC respectively. BX and BY intersect AC in P and Q respectively.

Prove that $AP=PQ=QC$.

Hint : Join BD intersecting AC at O.

P is the centroid of $\triangle ABD$. $OP=\frac{1}{3}AO$.

Q is the centroid of $\triangle BCD$. $OQ=\frac{1}{3}CO$.

□ □

16

AREA

16.1. AREA

In lower classes, you learnt how to find the areas of rectangles, squares, triangles and circles. Now we shall have a closer look at the concept of area of a plane figure.

By the **area** of a closed plane figure we mean the area of the region enclosed within its bounding lines.

The **unit of area** is the area of a square region whose side is a unit of length.

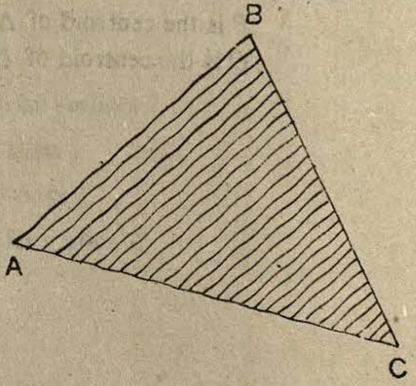
If the unit of length is a *metre* : the unit of area is a *square metre* ; if the unit of length is a *centimetre*, the unit of area is a *square centimetre*.

The terms *square decimetre*, *square decametre* and *square kilometre* are used in the same sense.

16.2. POLYGONAL REGIONS

We know that a triangle is a *closed plane figure* formed by three line segments. We are also familiar with the interior and the exterior of a triangle.

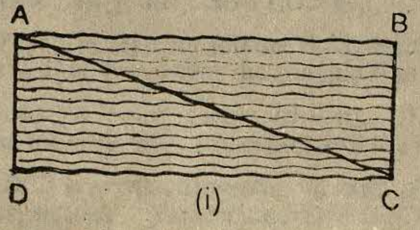
A triangular region is the union of a triangle and its interior.



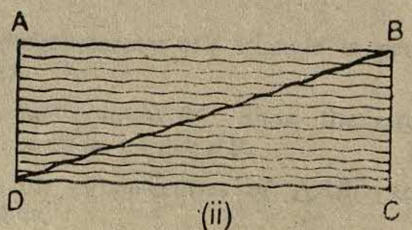
The shaded portion along with $\triangle ABC$ is triangular region *ABC*. It is a *part* of the plane enclosed by the $\triangle ABC$.

When we talk of the area of a triangle, we actually describe the magnitude of its triangular region.

Similarly, **a rectangular region is the union of a rectangle and its interior.**



(i)

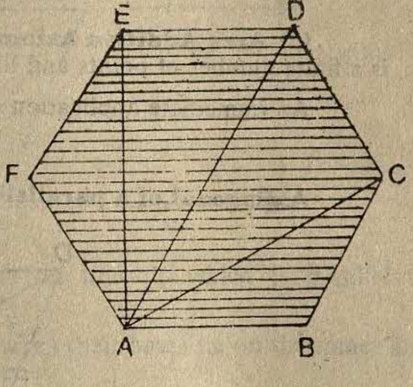


(ii)

Observe that the rectangular region *ABCD* can be expressed as the union of two triangular regions *ABC* and *ADC* as shown in the figure (*i*) or two triangular regions *ABD* and *BCD* as shown in the figure (*ii*).

A square region is the union of a square and its interior.

A polygonal region is the union of a finite number of triangular regions in a plane such that the intersection of any two of these triangular regions is either (i) empty or (ii) a point or (iii) a segment.



16.3. AREA OF A POLYGONAL REGION

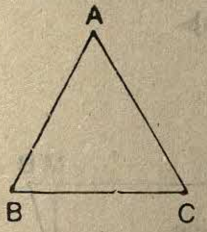
There is a close analogy between the concept and properties of the area of a polygonal region and those of the length of a line segment. We shall now discuss the corresponding axioms for the area of a polygonal region.

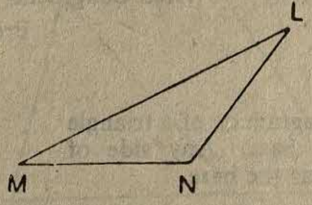
(1) The area of a polygonal region in square metres (sq. m. or m^2) or square cm (sq. cm or cm^2) is a **positive real number**.

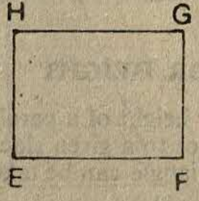
(2) The area of a polygonal region R is denoted by *ar* (R).

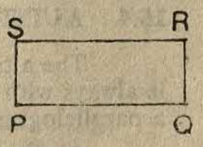
(3) **Area Congruence Axiom** : You have already learnt that two triangles are congruent if one can be placed on the other so as to coincide completely with it. When one triangle can be made to coincide with the other, the surface contained within the three sides of the one will coincide with the surface contained within the three sides of the other. Therefore, the two triangles are equal in area. Thus *congruent figures are equal in area*. But the converse of it is *not* necessarily true.

Study the following equivalent figures :









Place the $\triangle ABC$ over the $\triangle LMN$ and the rect. *EFGH* over the rect. *PQRS*.

What do you note ?

$\triangle ABC$ and $\angle LMN$ do not coincide with each other. Therefore, they are not congruent, but they are equal in area. Similarly, rectangles *EFGH* and *PQRS* are not congruent, but they are equal in area.

Thus equal figures are *not* necessarily congruent *i.e.*, figures which are not congruent may be equal in area.

If $\triangle ABC$ and $\triangle DEF$ are two congruent triangles, then,

$$ar\ (\text{region } \triangle ABC) = ar\ (\text{region } \triangle DEF)$$

Note. When the figures are *congruent* the sign '$\cong$' is used.

When the figures are *equal in area*, the sign '$=$' is used.

(4) If R_1 and R_2 are two polygonal regions such that

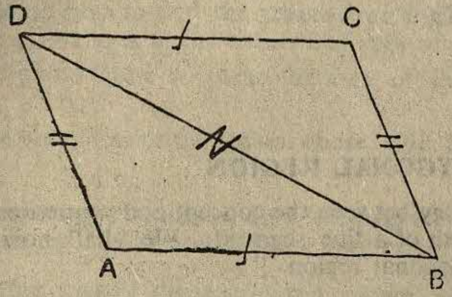
$$R_1 \subset R_2, \text{ then } ar\,(R_1) \leqslant ar\,(R_2)$$

(5) **Area Addition Axiom** : If R_1 and R_2 are two polygonal regions whose intersection is a finite number of points and line segments and $R = R_1 \cup R_2$, then $ar(R) = ar(R_1) + ar(R_2)$.

An immediate application of congruent area axiom is the following theorem.

THEOREM 32

A diagonal of a parallelogram divides it into two triangles of equal area.



Given : A parallelogram $ABCD$.

BD is one of the diagonals of parallelogram $ABCD$.

To prove : $ar\,(\triangle ABD) = ar\,(\triangle BCD)$

Proof : In	$\triangle ABD$ and $\triangle CDB$	
	$AB = CD$	[Opposite of a parallelogram]
	$DA = BC$	[Opposite of a parallelogram]
	$BD = BD$	[Common]
$\therefore$	$\triangle ABD \cong \triangle CDB$	[SSS—Congruency Theorem]
Hence	$ar\,(\triangle ABD) = ar\,(\triangle CDB)$	[Area Congruence Axiom]

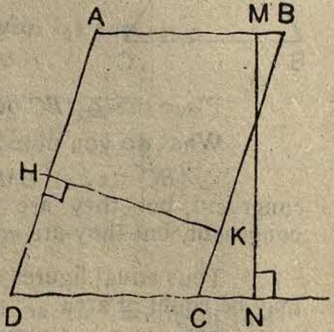
Proved

16.4. ALTITUDE OR HEIGHT

The altitude or height of a parallelogram or of a triangle is always with reference to a given side as base. Any side of a parallelogram, or triangle can be taken as the base.

The altitude of a parallelogram is the perpendicular distance between the base and the opposite side.

In parallelogram $ABCD$, if AD be regarded as base, HK is the altitude, and if DC be regarded as base, MN is the altitude.



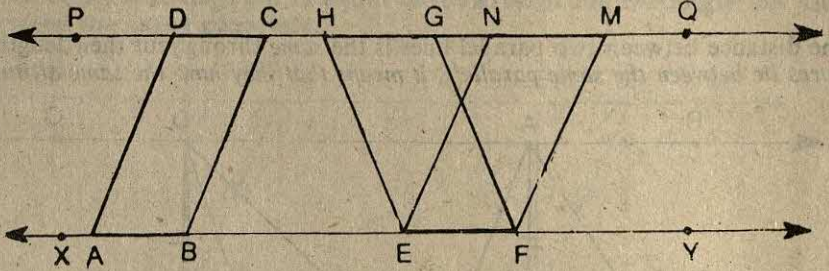
Note : (1) A parallelogram has *two* altitudes—one for each pair of parallel sides.

(2) A triangle has *three* altitudes—one for each vertex.

16.5. BETWEEN THE SAME PARALLELS

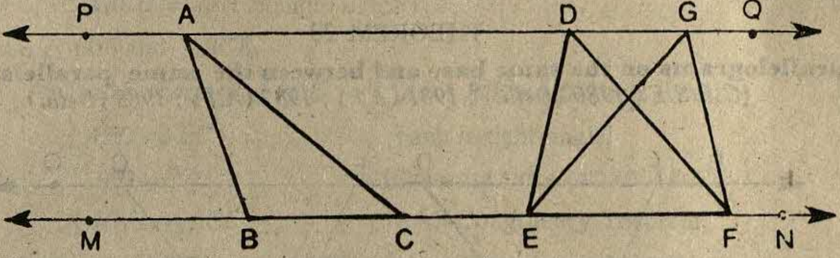
Parallelograms are said to be *between the same parallels* when their bases are in the same line and their sides opposite to these bases are also in another line parallel to the first.

$\|^{gms}$ *ABCD*, *EFGH* and *EFMN* are between the same parallels as their bases *AB*, *EF*, *EF*



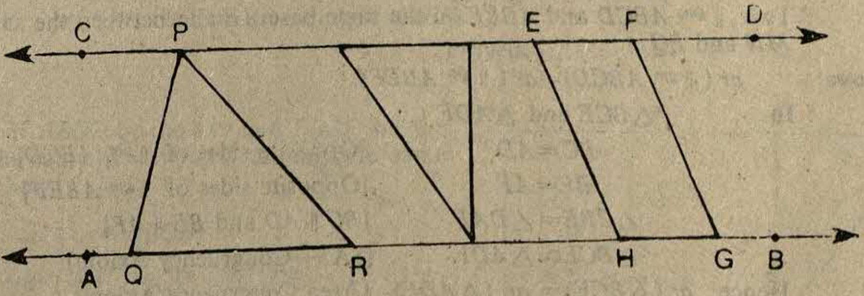
lie on line *XY* and their opposite sides *DC*, *HG*, *NM* lie on line *PQ* which is parallel to line *XY*.

Triangles are said to be *between the same parallels* when their bases lie on the same line and their opposite vertices lie on a line parallel to the first.

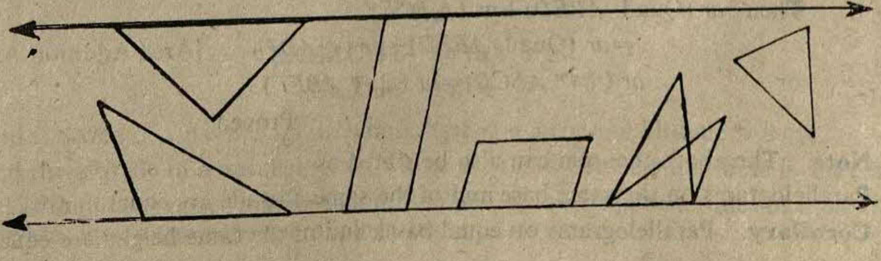


Δs *ABC*, *DEF* and *GEF* are between the same parallels because their bases *BC*, *EF*, *EF* are on the same line *MN* and their opposite vertices *A*, *D*, *G* are on the same line *PQ* which is parallel to *MN*.

In the figure below ΔPQR and $\|^{gm}$ *EHGF* are between the same parallels *AB* and *CD*.

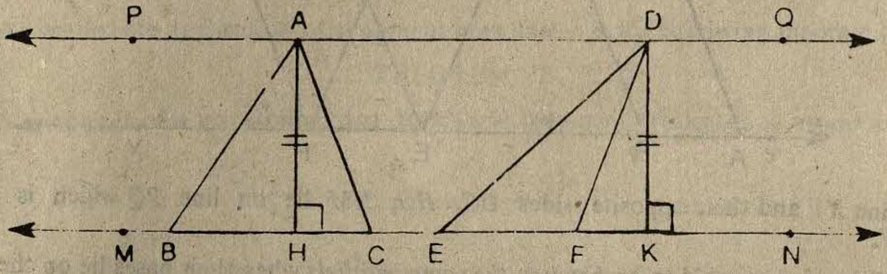


Examine the following figures :



These triangles and parallelograms are within the same parallel lines. They merely occupy some space anywhere between two parallel lines. They are *not* between the same parallels.

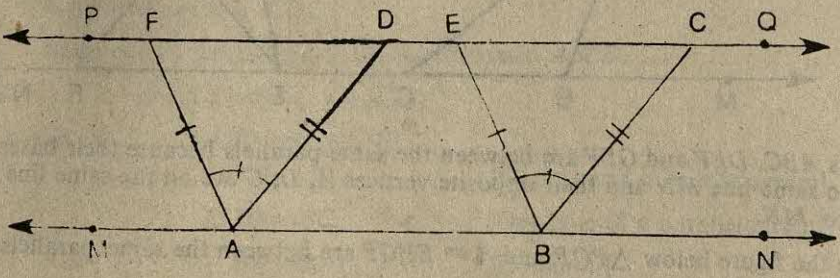
The distance between two parallel lines is the *same* throughout their lengths. Therefore, *if two figures lie between the same parallels, it means that they have the same altitude.*



In the above figure $\triangle$s ABC and DEF lie between the same parallels MN and PQ and hence their altitudes AH and DK are equal.

THEOREM 33

Parallelograms on the same base and between the same parallels are equal in area. [*C.B.S.E., 1980 (Delhi) ; 1981 (A.I.) ; 1982 (A.I) ; 1983 (Delhi) ; 1984 (Delhi)*]



Given : Two $\parallel^{gms}$ $ABCD$ and $ABEF$ on the same base AB and between the same parallels MN and PQ.

To prove : $ar(\parallel^{gm} ABCD) = ar(\parallel^{gm} ABEF)$

Proof : In $\triangle BCE$ and $\triangle ADF$

$BC = AD$ [Opposite sides of $\parallel^{gm}$ $ABCD$]

$BE = AF$ [Opposite sides of $\parallel^{gm}$ $ABEF$]

$\angle CBE = \angle DAF$ [$BC \parallel AD$ and $BE \parallel AF$]

$\therefore$ $\triangle BCE \cong \triangle ADF$ [SAS—Congruency Axiom]

Hence $ar(\triangle BCE) = ar(\triangle ADF)$ [Area Congruence Axiom]

Now ar (Quad. $ABED$) $= ar$ (Quad. $ABED$)

Then ar (Quad. $ABED$) $+ ar(\triangle BCE)$

$= ar$ (Quad. $ABED$) $+ ar(\triangle ADF)$ [Area Addition Axiom]

or $ar(\parallel^{gm} ABCD) = ar(\parallel^{gm} ABEF)$

Proved

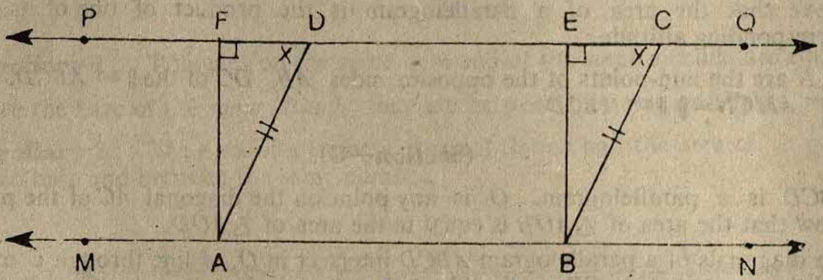
Note. The above theorem can also be stated as

Parallelograms on the same base and of the same altitude are equal in area.

Corollary. Parallelograms on equal bases and of the same height are equal in area.

THEOREM 34

The area of a parallelogram is equal to the area of the rectangle on the same base and between the same parallels.



Given : $\|^{gm}$ *ABCD* and rectangle *ABEF* on the same base *AB* and between the same parallels *MN* and *PQ*.

To prove : $ar(\|^{gm} ABCD) = ar(\text{rectangle } ABEF)$

Proof : In $\triangle ADF$ and $\triangle BCE$

$\angle ADF =$ corresp. $\angle BCE$ [*AD* ∥ *BC* and *DC* cuts them]

$\angle AFD = BEC$ [Each is right-angle]

$AD = BC$ [Opposite sides of $\|^{gm}$ *ABCD*]

$\therefore \quad \triangle ADF \cong \triangle BCE$ [AAS-Congruency Theorem]

Hence $ar(\triangle ADF) = ar(\triangle BCE)$ [Area Congruence Axiom]

Now ar (quad. *ABED*) $= ar$ (quad. *ABED*)

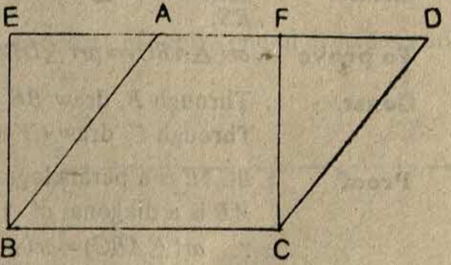
$\therefore ar$ (quad. *ABED*) $+ ar(\triangle BCE)$

$= ar$ (quad. *ABED*) $+ ar(\triangle ADF)$

[Area Addition Axiom]

i.e., $ar(\|^{gm} ABCD) = ar(\text{rectangle } ABEF)$

Proved.



Note. If *ABCD* be the $\|^{gm}$ and *EBCF* be the rectangle on the same base *BC* and having the same altitude *BE* ; then

$ar(\|^{gm} ABCD)$

$= ar(\text{rect. } EBCF) = BC.CF.$

$\therefore$ **area of a** $\|^{gm}$ **= base × altitude.**

EXERCISE—16 (a)

(Section—A)

1. (*a*) Find the area of a parallelogram whose base is 6·5 cm and altitude is 8 cm.

   (*b*) Find the altitude of a parallelogram whose area is 8·4 sq cm and base is 1·4 cm.

2. Prove that the line segment joining the mid-points of two opposite sides of a parallelogram divides it into two equal parallelograms.

3. In parallelogram $ABCD$, $AB=8$ cm. The altitudes corresponding to sides AB and AD are respectively 4 cm and 5 cm. Find the length of AD.

(Section—B)

4. Prove that the area of a parallelogram is the product of one of its sides and the corresponding altitude.

5. M, N are the mid-points of the opposite sides AB, DC of the $\parallel^{gm}$ $ABCD$. Prove that $\parallel^{gm}$ $AMCN=\frac{1}{2}$ $\parallel^{gm}$ $ABCD$.

(Section—C)

6. $ABCD$ is a parallelogram. O is any point on the diagonal AC of the parallelogram. Show that the area of $\triangle AOB$ is equal to the area of $\triangle AOD$.

7. The diagonals of a parallelogram $ABCD$ intersect in O. A line through O meets AB in X and the opposite side CD in Y. Show that

$$ar(\text{quad. } AXYD)=\tfrac{1}{2}\, ar(\parallel^{gm} ABCD)$$

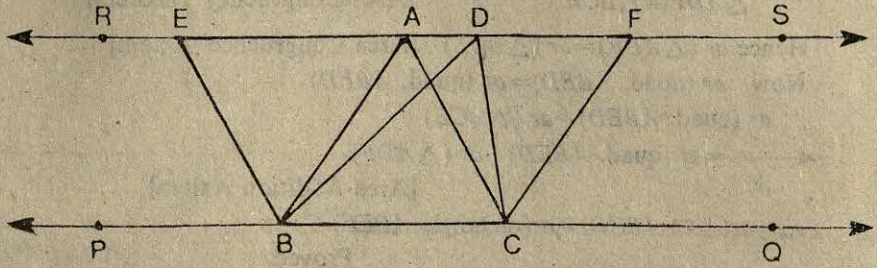
Hint : $\triangle AOX \cong \triangle COY$ Then $ar(\triangle AOX)=ar(\triangle COY)$

Add ar (quad $AOYD$) to each.

8. E and F are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$. Show that $ar(\triangle AEB)=ar(\triangle BFC)$. [*C.B.S.E., 1978 Delhi*) ; *1979 (A I.)*]

THEOREM 35

Triangles on the same base and between the same parallels are equal in area.



Given : $\triangle ABC$ and $\triangle DBC$ on the same base BC and between the same parallels PQ and RS.

To prove : $ar(\triangle ABC)=ar(\triangle DBC)$.

Const. : Through B, draw $BE \parallel CA$ intersecting line RS in E.
Through C, draw $CF \parallel BD$ intersecting line RS in F.

Proof : $BCAE$ is a parallelogram. [$BC \parallel EA$ and $BE \parallel CA$]

AB is a diagonal of $\parallel^{gm}$ $BCAE$.

$\therefore$ $ar(\triangle ABC)=ar(\triangle ABE)$

or $ar(\triangle ABC)=\frac{1}{2}\, ar(\parallel^{gm} BCAE)$

$BCFD$ is a parallelogram. [$BC \parallel DF$ and $CF \parallel BD$]

CD is a diagonal of $\parallel^{gm}$ $BCFD$.

$\therefore$ $ar(\triangle DBC)=\frac{1}{2}\, ar(\parallel^{gm} BCFD)$

$\parallel^{gm}$ $BCAE$ and $\parallel^{gm}$ $BCFD$ stand on the same base BC and are between the same parallels PQ and RS.

$\therefore$ $ar(\parallel^{gm} BCAE)=ar(\parallel^{gm} BCFD)$

or $\frac{1}{2}\, ar(\|^{gm}\, BCAE) = \frac{1}{2}\, ar(\|^{gm}\, BCFD)$

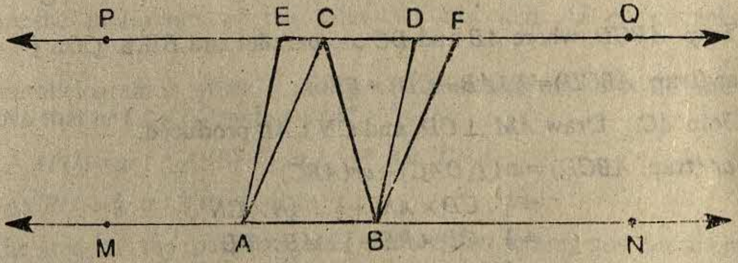
i e., $ar(\triangle ABC) = ar\,(\triangle DBC)$

Proved.

Corollary 1 : Triangles on the same base and of the same altitude are equal in area.

Since they are of the *same altitude,* they are between the same parallels.

Corollary 2 : The area of a triangle is equal to one half the area of a parallelogram on the same base and between the same paralles.



Given : $\triangle ABC$ and $\|^{gm}\, ABDE$ on the same base AB and between the same parallels MN and PQ.

To prove : $ar(\triangle ABC) = \frac{1}{2}\, ar(\|^{gm}\, ABDE)$

Const. : Complete the parallelogram $ABFC$.

Proof : BC is a diagonal of $\|^{gm}\, ABFC$.

$\therefore\ ar(\triangle ABC) = \frac{1}{2}\, ar(\|^{gm}\, ABFC)$

$\|^{gm}\, ABDE$ and $\|^{gm}\, ABFC$ are standing on the same base AB and are between the same parallels MN and PQ.

$\therefore\ ar(\|^{gm}\, ABDE) = ar(\|^{gm}\, ABFC)$

Hence $ar(\triangle ABC) = \frac{1}{2}\, ar(\|^{gm}\, ABDE)$.

Corollary 3 : The area of a triangle is half the product of only one of its sides and the corresponding altitude.

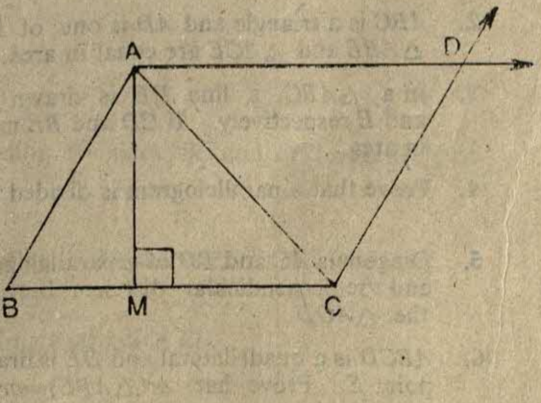
Hint : AM is the altitude to the side BC.

Draw $CD \parallel BA$ and $AD \parallel BC$ forming $\|^{gm}\, ABCD$

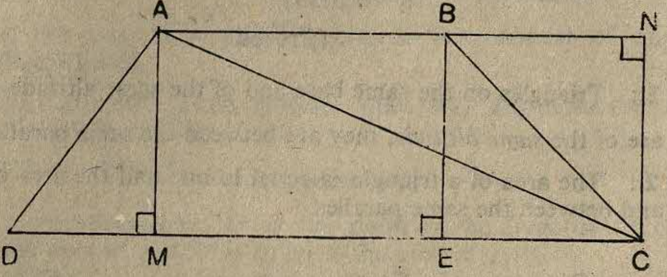
$ar(\triangle ABC) = \frac{1}{2}\, ar(\|^{gm}\, ABCD)$

$ar(\|^{gm}\, ABCD) = BC \times AM.$

$\therefore\ ar(\triangle ABC) = \frac{1}{2} \cdot BC \times AM$



Corollary 4 : The area of a trapezium is half the product of the height and the sume of the parallel sides.



Given : Trap. $ABCD$, where AB and DC are parallel and $BE \perp CD$.

To prove : ar (trap. $ABCD) = \frac{1}{2}(AB+CD) \times BE$

Const. : Join AC. Draw $AM \perp CD$, and $CN \perp AB$ produced.

Proof : ar (trap. $ABCD) = ar(\triangle DAC) + ar(ABC)$

$$= \tfrac{1}{2} \,.\, CD \times AM + \tfrac{1}{2} \,.\, AB \times CN$$

$$= \tfrac{1}{2} \,.\, CD \times BE + \tfrac{1}{2} \,.\, AB \times BE \qquad [AM = BE = CN]$$

$$= \tfrac{1}{2}(CD + AB) \times BE$$

$$= \tfrac{1}{2}(AB + CD) \times BE$$

EXERCISE—16 (b)

(Section—A)

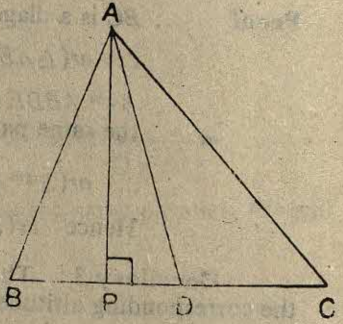
1. Show that a median of a triangle divides it into two triangles of equal area. [*C B.S.E., 1980 (Delhi) ; 1980 (A.I) ; 1982 (Dehli*]

Hint : Draw $AP \perp BC$.

$$ar\,(\triangle ABD) = \tfrac{1}{2} \,.\, BD \times AP$$

$$ar\,(\triangle ACD) = \tfrac{1}{2} \,.\, DC \times AP$$

$$= \tfrac{1}{2}\, BD \times AP$$



2. ABC is a triangle and AD is one of its medians. If E is any point on AD, show that $\triangle ABE$ and $\triangle ACE$ are equal in area.

3. In a $\triangle ABC$, a line DE is drawn parallel to BC intersecting AB and AC at D and E respectively. If CD and BE meet at O, prove that $\triangle BOD$ and $\triangle COE$ are equal in area.

4. Prove that a parallelogram is divided by its diagonals into four triangles of equal area. [*C.B.S.E , 1983 (Delhi)*]

5. Diagonals AC and BD of a parallelogram $ABCD$ intersect at O. Given that $AB = 12$ cm and the perpendicular distance between AB and DC is 6 cm, calculate the area of the $\triangle AOD$.

6. $ABCD$ is a quadrilateral and DE is drawn parallel to AC to intersect BC produced at a point E. Prove that $ar(\triangle ABE) = ar$(quad. $ABCD$) [*C.B.S.E., 1983 (A.I.)*]

(Section—B)

7. E and F are any two points lying on the side DC and AD respectively of a parallelogram $ABCD$.
Show that $ar\ (\triangle AEB) = ar\ (\triangle BFC)$.

8. ABC is a triangle in which D is the mid-point of BC and E is the mid-point of AD.
Prove that $ar\ (\triangle BED) = \frac{1}{4}\ (\triangle ABC)$.

•9. ABC is a triangle and D, E are the mid-points of AB, AC respectively) ; prove that $ar\ (\triangle ADE) = \frac{1}{4}\ ar\ (\triangle ABC)$.

(Section—C)

10. If D, E, F are the mid-points of the sides BC, CA and AB respectively, prove that $ar\ (\triangle DEF) = \frac{1}{4}\ ar\ (\triangle ABC)$.

11. $ABCD$ is a parallelogram in which P and Q are the middle points of BC and CD respectively. Prove that $ar\ (\triangle APQ) = \frac{3}{8}\ ar\ (\parallel^{gm} ABCD)$

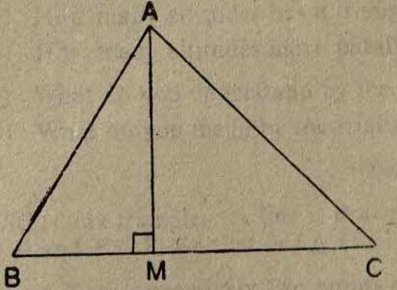
Hint : $ar\ (\triangle APQ) = ar\ (\triangle APB) = \frac{1}{4}\ ar\ (\parallel^{gm} ABCD)$

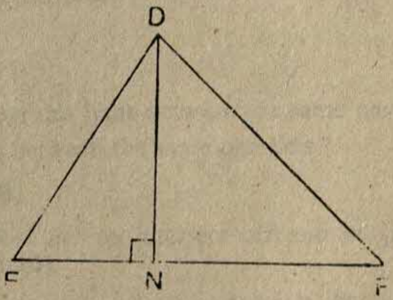
$ar\ (\triangle CPD) = \frac{1}{8}\ ar\ (\parallel^{gm} ABCD)$.

12. Prove that the area of the parallelogram formed by joining the middle points of the sides of a quadrilateral is half that of the quadrilateral.

THEOREM 36

Triangle having equal areas and having one side of one of the triangles equal to one side of the other, have their corresponding altitudes equal.





Given : $\triangle ABC$ and $\triangle DEF$ such that

$ar\ (\triangle ABC) = ar\ (\triangle DEF)$

and $BC = EF$

AM and DN are the altitudes corresponding to sides BC and EF respectively of the two triangles.

To Prove : $AM = DN$

Proof : In $\triangle ABC$, AM is the altitude corresponding to the side BC.

$\therefore \quad ar\ (\triangle ABC) = \frac{1}{2}\ BC \times AM$

In $\triangle DEF$, DN is the altitude corresponding to the side EF.

$\therefore \quad ar\ (\triangle DEF) = \frac{1}{2}\ .\ EF \times DN$

But $ar(\triangle ABC)=ar(\triangle DEF)$ [given]

Then $\frac{1}{2}.BC\times AM=\frac{1}{2}.EF\times DN$

$\therefore$ $BC\times AM=BC\times DN$ [$BC=EF$ given]

Hence $AM=DN$

Proved

EXERCISE—16(c)

(Section—A)

1. Prove that triangles having equal area and equal altitudes are on equal bases.

2. D and E are points on sides AB and AC respectively of $\triangle ABC$ such that

$$ar(\triangle BCE)=ar(\triangle BCD).$$

Prove that DE is parallel to BC.

3. $\triangle ABC$ and $\triangle DBC$ are on the same base BC with A and D on opposite sides of BC such that $ar(\triangle ABC)=ar(\triangle DBC)$. Show that BC bisects AD.

4. The bisectors of base angles B and C of an isosceles triangle ABC meet the sides in D and E. Prove that DE is parallel to BC.

(Section—B)

5. If each diagonal of a quadrilateral separates it into two triangles of equal area, then show that the quadrilateral is a parallelogram.

6. The diagonals AC and BD of quadrilateral $ABCD$ intersect at O and separate the quad. $ABCD$ into four triangles of equal area. Prove that quad. $ABCD$ is a parallelogram.

REVIEW EXERCISE XV

(Section—A)

1. Which of the following statements are True and which are False.

   (*a*) A rectangular region is the union of at least two triangular regions.

   (*b*) If two triangles are congruent, they have equal area.

   (*c*) Parallelograms between the same parallels are of the same altitude.

   (*d*) Triangles on equal bases and of equal altitudes are equal in area.

   (*e*) The area of a triangle is equal to the area of a parallelogram on the same base and between the same parallels.

   (*f*) The area of a parallelogram is equal to half the area of a rectangle on the sam base and between the same parallels.

2. Fill in the blanks in the following so as to make each of the statements true.

   (*a*) Parallelograms on the same base and between same ………… are equal in area. [*C.B.S.E., 1984* (*A.I.*)].

   (*b*) Triangles on the same base and between the same parallels are of ………… area.

   (*c*) A triangular region is the union of a triangle and its………… .

   (*d*) A rectangular region is the ………… of a rectangle and its interior.

   (*e*) A median of a triangle divides it into ………… triangles of equal area.

   (*f*) If two triangles are between the same parallels, they have the same ………… .

   (*g*) The area of a polygonal region in square units is a ………… real number.

   (*h*) Equal parallelograms on equal bases have the same ………… .

3. (*a*) How many altitudes has a triangle ?

   (*b*) How many altitudes has a parallelogram ?

4. (*a*) What do you understand by the parallelograms lying between the same parallels ?

   (*b*) What do you mean by the triangles lying between the same parallels ?

(Section—B)

5. ABC is any triangle. A line is drawn parallel to BC to intersect AB and AC produced in D and E. Prove that $ar\ (\triangle ABE) = ar(\triangle ACD)$.

6. D and E are respectively two points on sides AB and BC of a triangle ABC such that $DE \parallel AC$. Prove that $ar\ (\triangle ABE) = ar(\triangle DBC)$. [*C.B.S.E., 1977* (*A.I*) ; *1984 A.I.*)]

7. $ABCD$ is a trapezium in which $AB \parallel DC$. P is the mid-point of AD. Though P, RS is drawn parallel to BC. Prove that ar (trap. $ABCD) = ar\ (\parallel^{gm}\ BCRS)$.

8. $ABCD$ is a quadrilateral whose diagonals intersect in E. AC is produced to F such that $CF = AE$. Prove that $ar\ (\triangle BDF) = ar$ (quad. $ABCD$)

   Hint : $ar\ (\triangle DCF) = ar\ (\triangle DAE)$

   Also $ar\ (\triangle BCF) = ar\ (\triangle ABE)$

9. E is any point on diagonal AC of a parallelogram $ABCD$. Through E, RS is drawn parallel to AB intersecting AD in R and BC in S. Again through E, PQ is drawn parallel to BC intersecting AB in Q and DC in P.

   Prove that $ar\ (\parallel^{gm}\ REPD) = ar\ (\parallel^{gm}\ QBSE)$.

(Section—C)

10. Two triangles have two sides of the one equal to two sides of the other, each to each and the included angles are supplementary. Prove that the triangles are equal in the area.

11. *E, F, G* and *H* are respectively the mid-points of the sides *AB, BC, CD* and *DA* of a parallelogram *ABCD*. Show that quad. *EFGH* is a parallelogram and its area is half the area of the parallelogram *ABCD*.

12. A point *E* is taken on the side *CD* of a $\parallel^{gm}$ *ABCD* and *CD* is produced to *F* so that $DF = CE$. *BE* meets *AD* produced in *G* and the line through *F* parallel to *AG* in *H*.

Prove that

$ar(\parallel^{gm} ABCD) = ar(\parallel^{gm} AGHF)$.

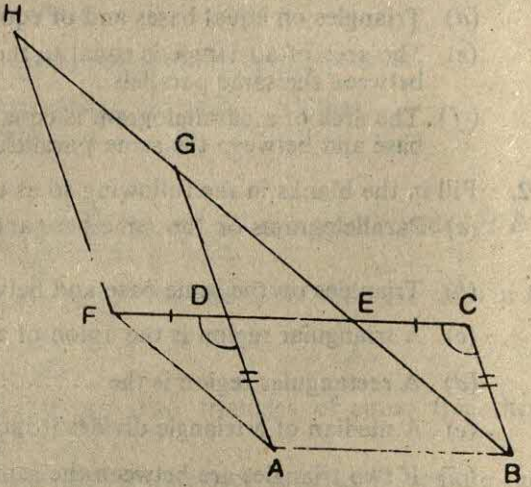
Hint : $\triangle DFA = \triangle CEB$

ABEF is a parallelogram.

$ar(\parallel^{gm} ABEF) = ar(\parallel^{gm} AGHF)$

Also $ar(\parallel^{gm} ABCD)$

$= ar(\parallel^{gm} ABEF)$.



17

GEOMETRICAL CONSTRUCTIONS

17.1. BASIC CONSTRUCTIONS

You have often drawn figures while proving a theorem or its riders. Graduated ruler, protractor, set-squares and compasses were freely used in the process. These figures were rough sketches. We shall now learn how to draw a correct and accurate figure from given data. These constructions are to be done with a straight ungraduated edge and compasses. But in our practical work we use a graduated ruler and compasses.

Every construction is completed in several steps which depend on one or more of the following basic constructions :

(1) **Line Construction**. It is possible to construct one and only one line through two given points.

(2) **Circle Construction**. It is possible to construct one and only one circle in a plane, with its centre at a given point in the plane and radius equal to the length of a given line segment.

(3) **Congruent Segment Construction**. Given a ray OX and a line segment AB, it is possible to mark a point P on ray OX such that $OP=AB$.

(4) **Circle Line Intersection Construction**. Given a point P outside a line l and a line segment AB, if the distance of P from l is less than AB, then it is possible to construct points X and Y on l such $PX=PY=AB$.

(5) **Two-Circle Intersection Construction** Given two points A and B and line segments of lengths r_1 and r_2, if $AB<r_1+r_2$, then it is possible to construct two points P and Q, one on either side of AB, such that $PA=QA=r_1$ and $PB=QB=r_2$.

Every construction consists of three parts.

(*i*) **Restatement**. The problem may be restated to specify clearly (*a*) given (*b*) required.

(*ii*) **Steps of Construction**. It is the sequence of steps that we actually take in the order in which they are taken to complete the construction.

(*iii*) **Proof**. It is like proving theorems We have to show that the constructed figure has the properties it was required to have.

The proof is an essential part of a construction. But, if it is not specifically required, it may not be provided.

For a neat and accurate figure of a construction, the following points should be observed :

(1) Always rule a line (line segment) from left to right.

(2) Draw given figure with moderate thick line segments.

Draw construction steps with fine segments.

Draw constructed figure with thick segments or arcs.

Auxilary lines needed for proof should be dotted.

(3) Pencil whether hard or soft, must be sharply pointed.

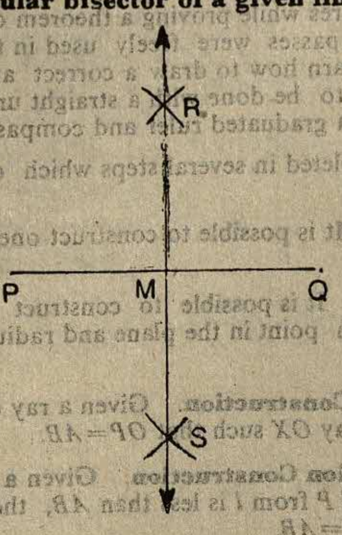
(4) Before using a compass, see that its two ends (the pencil end and pin end) meet at a point and the joint is firm.

(5) First draw a rough sketch figure and mark the given measurements on it. Then plan your steps.

17.2. ELEMENTARY CONSTRUCTIONS

CONSTRUCTION—1

To draw the perpendicular bisector of a given line segment.



Given : A line segment PQ.

Required : To draw the perpendicular bisector of PQ.

Steps of Construction :

(1) With P as centre and radius equal to more than $\frac{1}{2}PQ$, draw arcs on both sides of PQ.

(2) With Q as centre and the same radius draw arcs cutting the former arcs at R and S.

(3) Draw line RS and let it cut PQ at M.

Then line RS is the required perpendicular bisector of line segment PQ.

Proof : $PR=QR$ [Construction]

$\therefore$ R lies on the perpendicular bisector of PQ.

Also $PS=QS$ (Construction)

$\therefore$ S lies on the perpendicular bisector of PQ.

Thus, R and S lie on the perpendicular bisector of PQ.

Hence, line RS is the perpendicular bisector of line segment PQ.

Note that M is the mid-point of PQ *i.e.*, PQ is bisected at M.

By measurement $PM=MQ$.

EXERCISE—17 (*a*)

(Section—A)

1. Draw a line segment and draw its perpendicular bisector.
2. Given a line segment, find its mid-point, using ruler and compass.
3. Draw a line segment of length 5·6 cm and bisect it. Measure each part.

(Section—B)

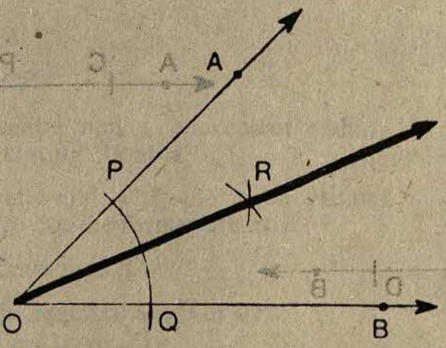
4. Draw a line segment 5·2 cm long and divide it into four equal parts. Measure the length of each part.
5. Draw any triangle. Find the mid-points of the sides. Join them to form a triangle.
6. Draw a line segment 4·8 cm long ; find by construction one-fourth part of it and verify by calculation.

(Section—C)

7. Find by construction the value of $\frac{5}{8}$ of 6·4 cm.

CONSTRUCTION—2

To draw the bisector of a given angle.



Given : An angle *AOB*.

Required : To draw the bisector of $\angle AOB$.

Steps of Construction :

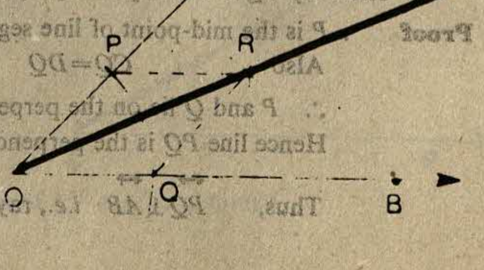
(1) With *O* as centre and a convenient radius draw an arc cutting the arm *OA* at *P* and the arm *OB* at *Q*.

(2) With centre *P* and a radius greater than half of *PQ*, draw an arc.

(3) With centre *Q* and the same radius, draw another arc to cut the former arc at *R*.

(4) Draw ray *OR*.

Then ray *OR* is the required bisector of $\angle AOB$.



Proof : Join *PR* and *QR*.

In $\triangle OPR$ and $\triangle OQR$

$OP = OQ$

$PR = QR$

$OR=OR$

$\therefore \quad \triangle OPR \cong \triangle OQR$ [SSS-Congruency Theorem]

Then $\angle POR = \angle QOR$

i.e. $\angle AOR = \angle BOR$

Hence, ray OR bisects $\angle AOB$.

EXERCISE—17 (*b*)

(Section—A)

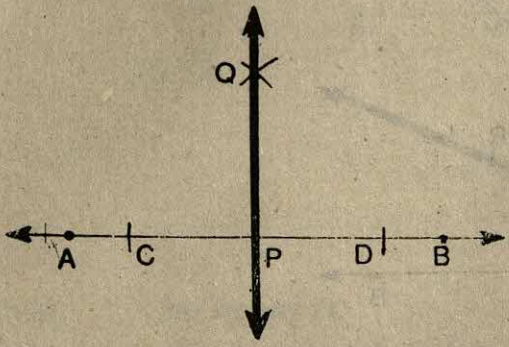
1. Draw an angle of 70° using the protector and bisect it.
2. Draw (*a*) an obtuse angle, (*b*) a right angle and bisect each of them.

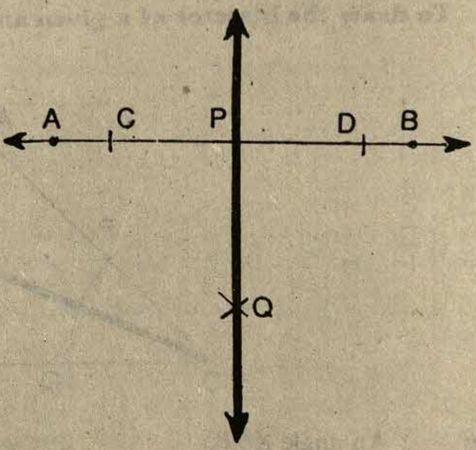
(Section—B)

3. Draw a line PQ. At a point O on it, draw ray OR. Bisect the angles POR and QOR. Measure the angle between the bisectors.

CONSTRUCTION—3

To draw a perpendicular to a given line at a point of it.





Given : A line AB and a point P on it.

Required : To draw a perpendicular to line AB at P.

Steps of Construction :

(1) With P as centre and any radius draw arcs to cut AB at C and D.

(2) With C as centre and radius more than half of CD, draw an arc.

(3) With D as centre and the same radius draw another arc to cut the former arc at Q,

(4) Draw a line through P and Q.

Then ray PQ is the required perpendicular to line AB at P.

Proof : P is the mid-point of line segment CD (Construction)

Also $CQ=DQ$ (Construction)

$\therefore$ P and Q lie on the perpendicular bisector of CD

Hence line PQ is the perpendicular bisector of CD.

Thus, $\overleftrightarrow{PQ} \perp \overleftrightarrow{AB}$ *i.e.*, ray PQ is perpendicular to line AB at P on it.

EXERCISE—17(c)

(Section—A)

1. Draw a line segment AB of length 5·5 cm. Mark a point P on it. Through P, draw a line perpendicular to AB.
2. Draw a line segment AB of length 6·3 cm. At a point P on AB, 2·4 cm from B, draw a ray PQ perpendicular to it.

(Section—B)

3. Draw a line segment AB of length 4·7 cm. At either end, draw a line perpendicular to AB.

   Are the two lines drawn parallel ?
4. Draw a line segment. At each end, draw a ray perpendicular to the line segment—one upward and the other downward.

   Are the two rays drawn parallel ?

With the help of above constructions, we can now construct angles of measure 60°, 30°, 120°, 90°, 45°, etc., using a ruler and compasses.

CONSTRUCTION—4 (*a*)

To construct an angle of 60° without using the protector.

Steps to Construction :

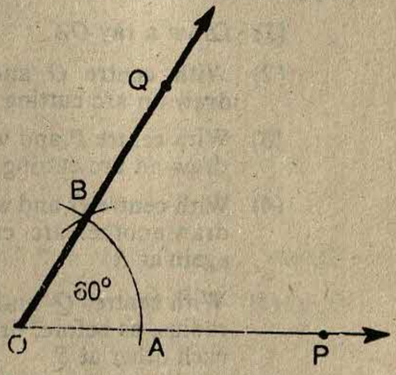
(1) Draw a ray OP.

(2) With O as centre and a convenient radius, draw an arc cutting OP at A.

(3) With A as centre and with the same radius, draw an arc to cut the former arc at B.

(4) Through B, draw ray OQ.

Then $\angle POQ$ is the required angle of 60°.



Proof : In $\triangle AOB$, $OA=OB=AB$ [Construction]

$\therefore$ $\triangle AOB$ is equilateral.

Hence $\angle POQ=\angle AOB=60°$

CONSTRUCTION—4(*b*)

To construct an angle of 30°.

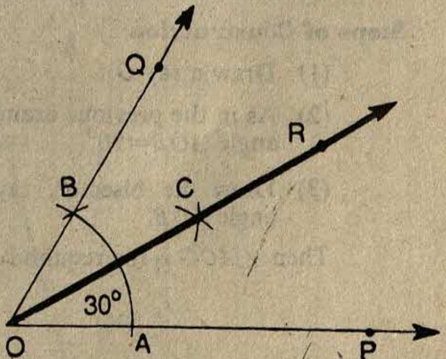
Steps of Construction :

(1) Draw a ray OP.

(2) As in the previous problem 4(*a*) make an angle $POQ=60°$.

(3) Draw the bisector, ray OC of the angle POQ.

Then $\angle POR$ is the reqd. angle of 30°,



Note. By bisecting the angle of 30°, an angle of 15° can be obtained.

CONSTRUCTION—4(*c*)

To construct an angle of 120°.

Steps of Costruction :

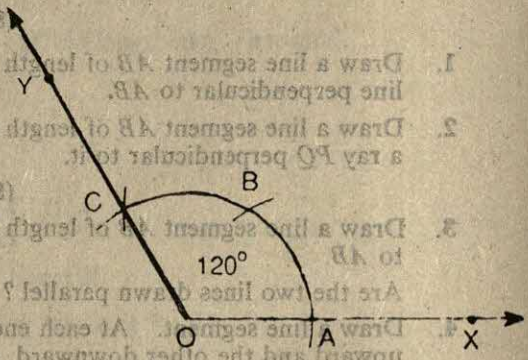
(1) Draw a ray *OX*.

(2) With *O* as centre and any convenient radius, draw an arc cutting *OX* at *A*.

(3) With *A* as centre and same radius, draw arcs cutting the former arc successively at *B* and *C*.

(4) Join *OC* and draw ray *OY* containing it.

Then ∠*XOY*=120°.



CONSTRUCTION—(*d*)

To construct an angle of 90°.

Steps of Construction :

(1) Draw a ray *OA*.

(2) With centre *O* and with any radius, draw an arc cutting *OA* at *P*.

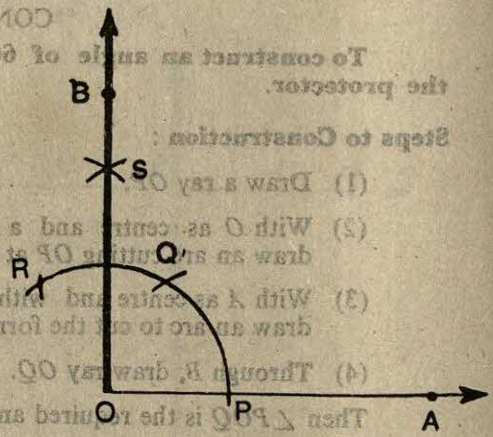
(3) With centre *P* and with the same radius, draw an arc cutting the first arc at *Q*.

(4) With centre *Q* and with the same radius, draw another arc cutting the first arc again at *R*.

(5) With centres *Q* and *R* and the same radius, as before, draw arcs intersecting each other at *S*.

(6) Join *OS* and draw ray *OB* containing it.

Then ∠*AOB* is the required angle of 90°.



CONSTRUCTION—4(*e*)

To construct an angle of 45°

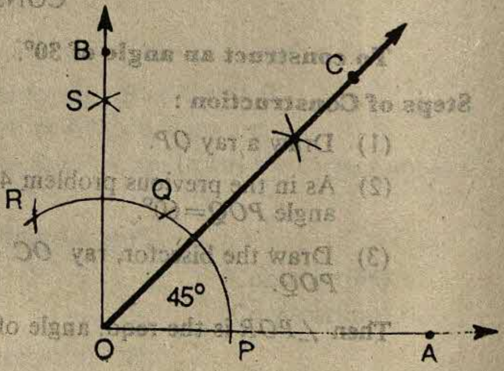
Steps of Construction :

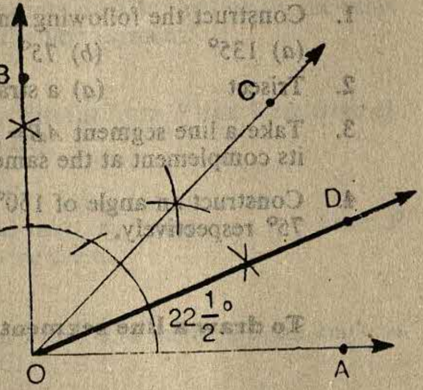
(1) Draw a ray *OA*.

(2) As in the previous example, make an angle *AOB*=90°.

(3) Draw the bisector, ray *OC* of the angle *AOB*.

Then ∠*AOC* is the required angle of 45°.



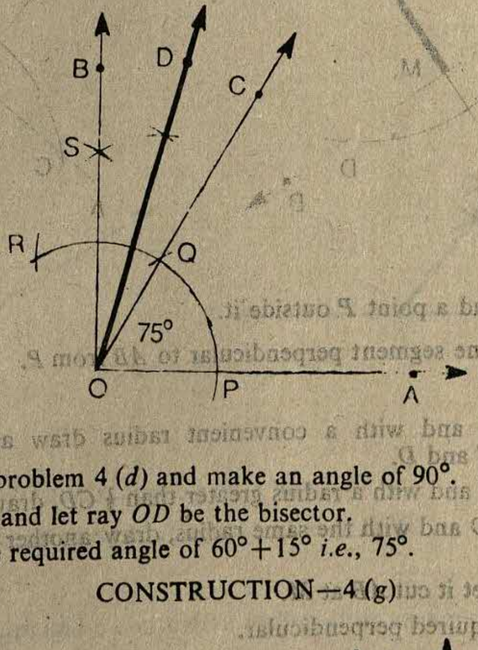


Note. By bisecting the angle of 45°, an angle of $22\frac{1}{2}°$ can be obtained.

$\angle AOD = 22\frac{1}{2}°$.

CONSTRUCTION—4(*f*)

To construct an angle of 75°.



Steps of Construction :

(1) Proceed as in problem 4 (*d*) and make an angle of 90°. [$\angle AOB = 90°$]

(2) Bisect $\angle BOC$ and let ray OD be the bisector.

Then $\angle AOD$ is the required angle of $60° + 15°$ *i.e.*, 75°.

CONSTRUCTION—4 (g)

To construct on angle of 105°.

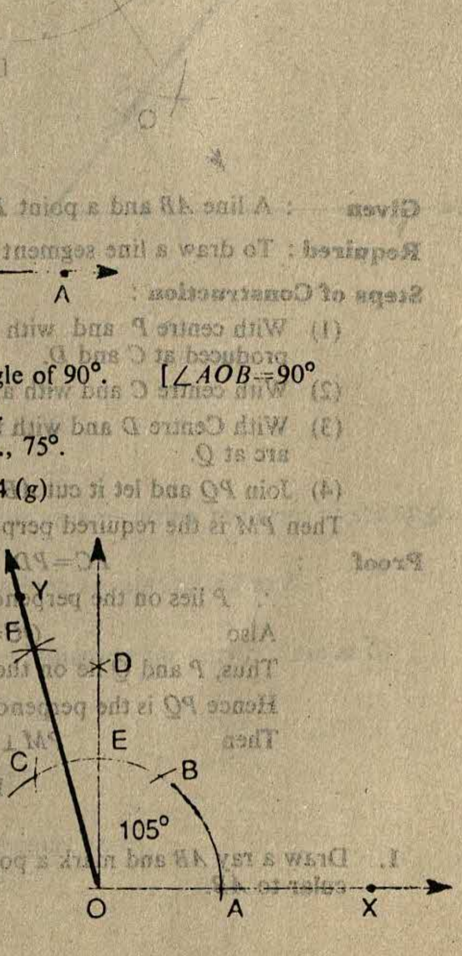
Steps of Construction :

(1) Proceed as problem 4 (*c*) and make angle of 120°.

(2) Bisect the $\angle BOC$ and let ray OD be the bisector.

(3) Again bisect the $\angle DOC$ by ray OF or ray OY.

Then $\angle XOY$ is the reqd. angle of 105°.

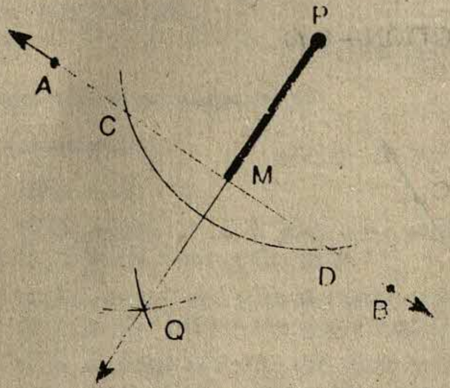


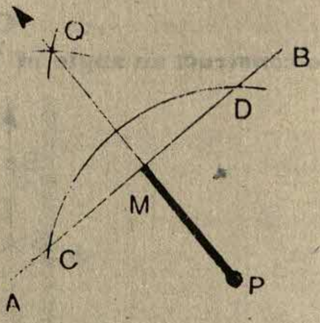
EXERCISE—17(c)

1. Construct the following angles with ruler and compass only :
   (*a*) 135° (*b*) 75° (*c*) $112\frac{1}{2}$° (*d*) 15°
2. Trisect (*a*) a straight angle, (*b*) a right angle.
3. Take a line segment $AB = 3{\cdot}4$ cm long. At *A* construct an angle of 30°. Then draw its complement at the same point *A*.
4. Construct an angle of 150° and divide it into three parts which measure 30°, 45° and 75° respectively.

CONSTRUCTION—5

To draw a line segment perpendicular to a given line from a point outside it.





Given : A line *AB* and a point *P* outside it.

Required : To draw a line segment perpendicular to *AB* from *P*.

Steps of Construction :

(1) With centre *P* and with a convenient radius draw an arc cutting *AB* or *AB* produced at *C* and *D*.

(2) With centre *C* and with a radius greater than $\frac{1}{2}$ *CD*, draw an arc.

(3) With Centre *D* and with the same radius, draw another arc cutting the previous arc at *Q*.

(4) Join *PQ* and let it cut *AB* at *M*.

Then *PM* is the required perpendicular.

Proof : $PC = PD$ (Construction)

$\therefore$ *P* lies on the perpendicular bisector of *CD*.

Also $QC = QD$ (Construction)

Thus, *P* and *Q* lie on the perpendicular bisector of *CD*.

Hence *PQ* is the perpendicular bisector of *CD*.

Then $PM \perp CD$ *i.e.,* $PM \perp AB$.

EXERCISE—17(d)

(Section—A)

1. Draw a ray *AB* and mark a point *P* outside it. Construct another ray *PQ* perpendicular to *AB*.

(Section—B)

2. Draw segment line segment 3.7 cm long. Take two points outside the line on the same side of it. Draw perpendicular from these points to the line segment.

   Are they parallel ?

3. *PQ* is a line segment 4.3 cm long. A point *R* is 6 cm distant from *P* and 7 cm from *Q*. Find by drawing and measurement how far *R* is from *PQ*.

(Section—C)

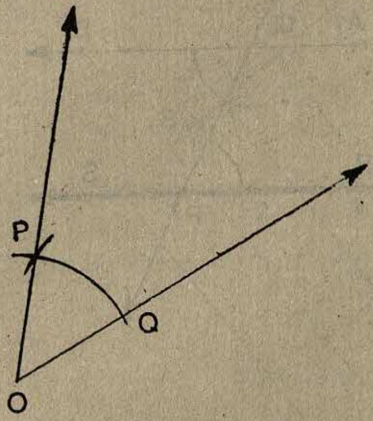
4. Construct an angle of 60°. Draw its bisector and take any point *P* on it. From *P* draw perpendiculars to the arms of the angle. Measure them.

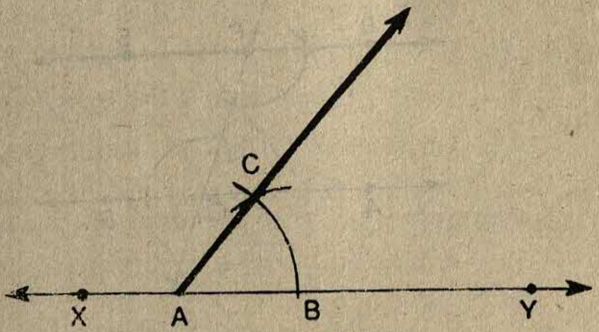
   What do you note ?

5. Draw two line segments *AB* and *AC*, 2·4 cm and 3·8 cm long and inclined at an angle of 45°. Join *BC* and draw a perpendicular from *A* to *BC*.

CONSTRUCTION 6

To construct an angle equal to given angle (at point on a given ray).





Given : An angle *POQ* and a point *A* on a line *XY*.

Required : To construct an angle at *A* equal to the ∠*POQ*.

Steps of Construction :

(1) With *O* as centre and a convenient radius draw an arc cutting the arms of the angle at *P* and *Q*.

(2) With *A* as centre and the same radius draw a long arc cutting *AY* at *B*.

(3) Take the distance *PQ* in the compasses.

(4) With *B* as centre and with the distance *PQ* as radius cut the pervious arc at *C*.

(5) Draw ray *AC*.

Then ∠*YAC* is the required angle.

EXERCISE—17(e)

(Section—A)

1. Draw an obtuse angle. Construct another angle equal to it.
2. Construct an angle equal to half of a given angle.

(Section—B)

3. Construct an angle equal to the sum of two given angles A and B.

4. At a given point in a given line make an angle equal to the supplement of a given angle.

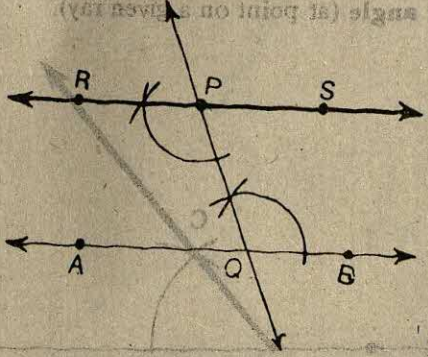
5. With your protractor, draw an angle of 80°. Now, with ruler and compass draw an angle of.

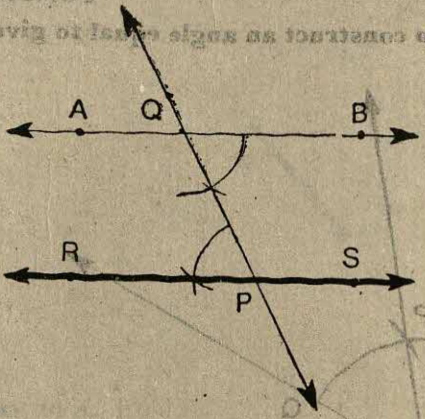
(*a*) 80° (*b*) 40° (*c*) 40° (*d*) 120°

CONSTRUCTION 7

To draw a line through a given point parallel to a given line (using the straight edge and the compasses).

In may be recalled that when a line cuts two other lines making apair of alternate angles equal, the lines are parallel. This property is made use of in this method of drawing parallel st. lines.





Given : A line AB, and a point P outside it.

Required : To draw a line through P parallel to AB.

Steps of Construction :

(1) Through P draw a line PQ cutting AB at Q.

(2) At P and on the other side of PQ make an angle QPR equal to $\angle BQP$.

(3) Draw line RP.

Then RS is the required parallel line through P.

EXERCISE—17 (f)

(Section—A)

1. Through a point outside a line how many lines can be drawn parallel to the line ?

(Section—B)

2. Draw a line parallel to a given line through a given point by making corresponding angles equal.

3. Draw a line PQ parallel to a given line XY at a distance of 2 cm from it.

4. AB is a given line. Mark 5 points on the same side of AB, so that each point is at the same distance of 3 cm from AB.

5. *PQ* and *RS* are two parallel lines.
   - (*a*) Draw a line segment which represents the distance between the parallels.
   - (*b*) Mark a point which is midway between them.
   - (*c*) Mark four other points which are equidistant from both the given lines.

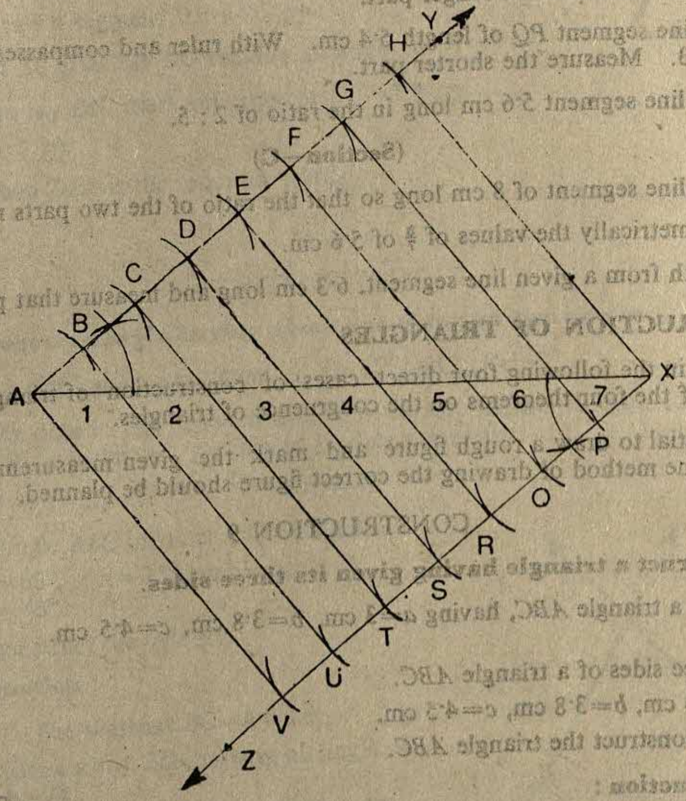
CONSTRUCTION—8

To divide a given line segment into a number of equal parts.

Divide a line segment 8·4 cm long into seven equal parts.

Given : A line segment *AX* 8·4 cm long.

Required : To divide it into 7 equal parts.



Const. : (*i*) At *A*, make an acute $\angle XAY$.

(*ii*) From *X*, draw ray *XZ* on the other side, making $\angle AXZ = \angle XAY$.

(*iii*) Along *AY* step off the required number (7) of equal distances successively and name the points of division *B, C, D, E, F, G, H*.

(*iv*) Similarly along *XZ* step off the same distances and name the points *P, Q, R, S, T, U, V*.

(*v*) Join *AV*, *BU*, *CT*, *DS*, *ER*, *FQ*, *GP* and *HX*.

These segments divide *AX* into the required number of equal parts.

EXERCISE—17 (f)

(Section—A)

1. Draw a line segment AB of length 4 cm. With ruler and compasses divide AB into three equal parts.
2. Draw a line segment PQ of length 6 cm. With ruler and compasses divide PQ into five equal parts.
3. Draw a line of length 43 mm and divide it into five equal parts.

(Section—B)

4. Draw a line segment AB of length 5 cm. With ruler and compasses divide AB in the ratio 1 : 2. Measure the longer part.
5. Draw a line segment PQ of length 5·4 cm. With ruler and compasses divide PQ in the ratio 2 : 3. Measure the shorter part.
6. Divide a line segment 5·6 cm long in the ratio of 2 : 5.

(Section—C)

7. Divide a line segment of 8 cm long so that the ratio of the two parts may be $\frac{3}{7}$.
8. Find geometrically the values of $\frac{3}{7}$ of 5·6 cm.
9. Cut off $\frac{2}{9}$th from a given line segment, 6·3 cm long and measure that part.

17.3. CONSTRUCTION OF TRIANGLES

The data in the following four direct cases of construction of triangles correspond to the hypothesis of the four theorems on the congruence of triangles.

It is essential to draw a rough figure and mark the given measurements. Then, with reference to it, the method of drawing the correct figure should be planned.

CONSTRUCTION 9

To construct a triangle having given its three sides.

Construct a triangle ABC, having $a=3$ cm, $b=3·8$ cm, $c=4·5$ cm.

Given : Three sides of a triangle ABC.
$a=3$ cm, $b=3·8$ cm, $c=4·5$ cm.

Required : To construct the triangle ABC.

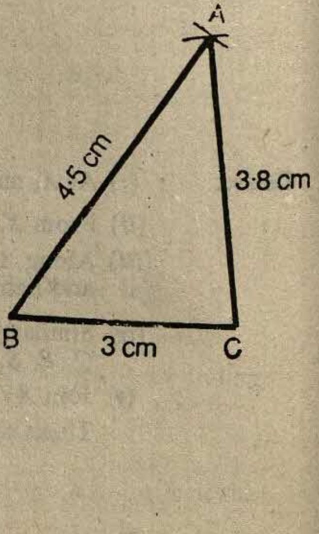
Steps of Construction :

(1) Draw a line segment $BC=3$ cm.

(2) With centres B and C and with radii equal to 4·5 cm and 3·8 cm respectively draw arcs intersecting each other at A.

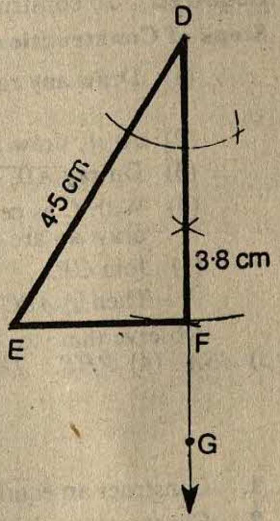
(3) Join AB and AC.

Then ABC is the required triangle.



CONSTRUCTION 10

To construct a triangle having given two sides and the included angle.



Construct a $\triangle DEF$ in which $DE=4{\cdot}5$ cm, $DF=3{\cdot}8$ cm and $\angle D=30°$.

Given : A triangle DEF in which $DE=4{\cdot}5$ cm, $DF=3{\cdot}8$ cm and $\angle EDF=30°$.

Required : To construct the triangle DEF.

Steps of Construction :

(1) Draw a segment $DE=4{\cdot}5$ cm.

(2) At D, draw ray DG making an angle of 30°.

(3) On ray DG, mark off segment $DF=3{\cdot}8$ cm.

(4) Join EF.
Then DEF is the required triangle.

CONSTRUCTION 11

To construct a triangle having given one side and two angles.

Construct $\triangle ABC$, given $a=4{\cdot}4$ cm, $\angle A=75°$ and $\angle B=60°$.

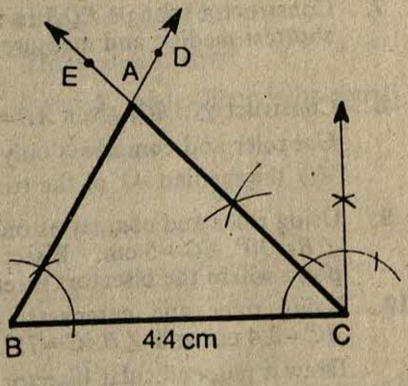
Compare the data in this problem with the corresponding theorem on the congruence of triangles. It will be clear that $\angle C$ has to be determined first.

$$\angle C=180°-(A+B)=180°-(75°+60°)=45^c$$

Given : A triangle ABC in which $BC=4{\cdot}4$ cm, $\angle B=60°$, $\angle A=75°$, and hence $\angle C=45°$.

Required : To construct the $\triangle ABC$.

Steps of Construction :



(1) Draw a line segment $BC=4{\cdot}4$ cm.

(2) At B, draw a ray BD making an angle of 60° with BC.

(3) At C, draw ray CE making $\angle BCE=45°$.

(4) Let ray BD and ray CE intersect at A.
Then ABC is the required triangle.

CONSTRUCTION 12

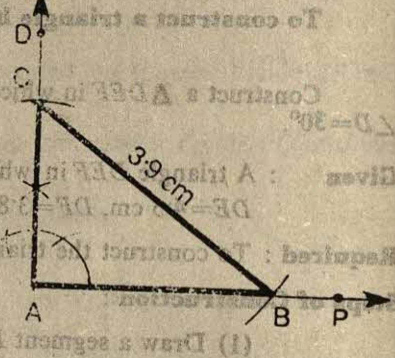
To construct a right-angled triangle in which the hypotenuse and one side are given.

Construct a right-angled triangle in which the hypotenuse$=3{\cdot}9$ cm, and one side$=2{\cdot}4$ cm.

Given : A triangle ABC in which $\angle A = 1$ rt. $\angle$, hyp. $BC = 3{\cdot}9$ cm, side $AC = 2{\cdot}4$ cm.

Required : To construct the triangle ABC.

Steps of Construction :



(1) Draw any ray AP.

(2) At A, draw $\overrightarrow{AD} \perp \overrightarrow{AP}$.

(3) On ray AD, cut off segment $AC = 2{\cdot}4$ cm.

(4) With C as centre and with radius $3{\cdot}9$ cm draw an arc cutting ray AP at B.

(5) Join CB.

Then $\triangle ABC$ is the required right-angled triangle.

Observe that to construct a triangle, we need three of its parts— (1) *SSS* (2) *SAS* (3) *ASA* (4) *RHS*. *There is no unique triangle, if only three angles are given.*

EXERCISE 17(g)

(Section—A)

1. Construct an equilateral triangle whose one side is $2{\cdot}6$ cm.
2. Construct a right-angled triangle whose longest side is $5{\cdot}6$ cm and another side is $3{\cdot}2$ cm.
3 Construct a triangle ABC having $AB = 8$ cm, $BC = 6$ cm and $CA = 5$ cm.

(Section—B)

4. Construct a triangle ABC, in which $a = 4{\cdot}6$ cm, $\angle B = 45^\circ$, $\angle A = 105^\circ$. Draw the longest altitude.
5. Draw a triangle ABC, having given $\angle C = 45^\circ$, $c = 2{\cdot}9$ cm and $a = 3{\cdot}7$ cm. Can more than one triangle have these parts ? If so, measure their other parts.
6. Construct an isosceles triangle with base $2{\cdot}8$ cm and vertical angle $= 30^\circ$.
7. Construct a triangle PQR in which $PQ = 3$ cm, RP $4{\cdot}6$ cm and $\angle Q = 135^\circ$. Draw the shortest median and measure it.

(Section—C)

8. Construct $\triangle ABC$, given $AB = 3{\cdot}5$ cm, $BC = 4{\cdot}5$ cm, and $AC = 5{\cdot}5$ cm. Use ruler and compasses only to construct.
   (*i*) the median AX of the triangle, (*ii*) the altitude from B to AC.
9. Using ruler and compasses only, construct a triangle ABC having given $\angle C = 135^\circ$, and $\angle B = 30^\circ$, $BC = 5$ cm. Bisect angles B and C and measure the distance of A from the point where the bisectors meet.
10. Using ruler and compasses only, construct a triangle ABC, having $AB = 4{\cdot}7$ cm, $AC = 3{\cdot}4$ cm and $\angle BAC = 75^\circ$.
    Draw a perpendicular bisector of BC and the bisector of $\angle BAC$. If perpendicular bisector and the bisector meet at point M, measure the $\angle BMC$.

17.4. CONSTRUCTION OF TRIANGLES FROM OTHER DATA

CONSTRUCTION 13

To construct a triangle having given two sides and the altitude to the third side.

Construct a triangle ABC having given $AB = 3{\cdot}2$ cm, $AC = 2{\cdot}6$ cm and the $\perp$ from A to $BC = 2{\cdot}2$ cm.

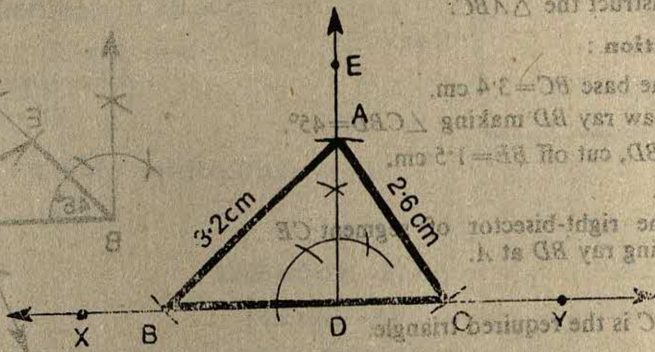
Draw a rough figure and analyse. A little thought will suggest that the base line XY and the perp. AD are to be drawn first. Then B and C can be determined by drawing arcs with A as centre and with radii equal to the given lengths 3·2 cm and 2·6 cm.

Steps of Construction :

(1) Draw any line XY.

(2) From any point D on it draw a perp. DE.

(3) From it, cut off DA equal to 2·2 cm.



(4) With A as centre and radii equal to 3·2 cm and 2·6 cm draw arcs cutting XY at B and C respectively.

(5) Join AB and AC.

Then ABC is the required triangle.

CONSTRUCTION 14

To construct a triangle, having given the base, one base angle and the sum of the other two sides.

Construct a $\triangle ABC$ in which $BC=3{\cdot}6$ cm, $AB+AC=4{\cdot}8$ cm and $\angle B=60°$.

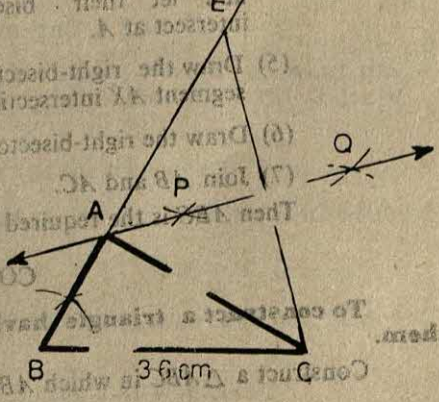
Draw a rough figure. Mark the given measurements. Think and analyse.

Since $BA+AC=BE$. A must lie somewhere between B and E, and AE must be equal to AC.

Given : A triangle ABC in which $a=3{\cdot}6$ cm, $\angle ABC=60°$ and $b+c=4{\cdot}8$ cm.

Required : To construct the $\triangle ABC$.

Steps of Construction :



(1) Draw the given base $BC=3{\cdot}6$ cm.

(2) At B, draw ray BD making $\angle CBD=60°$.

(3) On ray BD, cut off $BE=4{\cdot}8$ cm. (*i.e.*, the sum of the other two sides).

(4) Join CE.

(5) Draw the perpendicular bisector of segment CE and let it cut BE at A.

(6) Join AC.

Then $\triangle ABC$ is the required triangle.

CONSTRUCTION 15

To construct a triangle, having given the base, one base angle and the difference of the other two sides.

Construct a $\triangle ABC$, having given $\angle B = 45^\circ$, $a = 3{\cdot}4$ cm, and $c - b = 1{\cdot}5$ cm.

Given : A triangle ABC in which $BC = 3{\cdot}4$ cm, $\angle B = 45^\circ$ and $c - b = 1{\cdot}5$ cm.

Required : To construct the $\triangle ABC$.

Steps of Construction :

(1) Draw the base $BC = 3{\cdot}4$ cm.

(2) At B, draw ray BD making $\angle CBD = 45^\circ$.

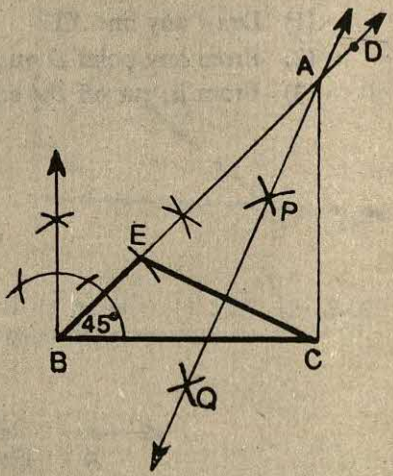
(3) On ray BD, cut off $BE = 1{\cdot}5$ cm.

(4) Join CE.

(5) Draw the right-bisector of segment CE intersecting ray BD at A.

(6) Join AC.

Then ABC is the required triangle.



CONSTRUCTION 16

To construct a triangle having given its perimeter and the angles at the base.

Construct a triangle whose perimeter is 6·4 cm, and angles at the base are 60° and 45°.

Given : The perimeter of a $\triangle ABC$ is 6·4 cm, $\angle B = 60^\circ$ and $\angle C = 45^\circ$.

Required : To construct the triangle ABC.

Const. : (1) Draw a line segment XY, 6·4 cm long.

(2) At X, make $\angle YXP = 60^\circ$.

(3) At Y, make $\angle XYQ = 45^\circ$.

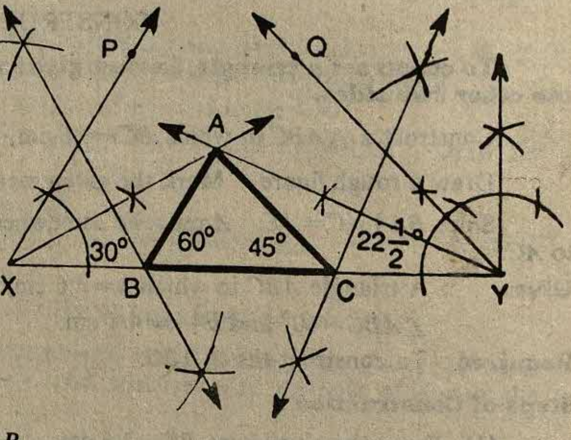
(4) Bisect these two angles and let their bisectors intersect at A.

(5) Draw the right-bisector of segment AX intersecting XY at B.

(6) Draw the right-bisector of segment AY intersecting XY at C.

(7) Join AB and AC.

Then ABC is the required triangle.



CONSTRUCTION 17

To construct a triangle having given its two sides and the median between them.

Construct a $\triangle ABC$ in which $AB = 3{\cdot}2$ cm, $BC = 4{\cdot}8$ cm and median $BD = 2{\cdot}8$ cm.

Given : *A* triangle *ABC* in which $AB=3{\cdot}2$ cm, $BC=4{\cdot}8$ cm and median $BD=2{\cdot}8$ cm.

Required : To construct the $\triangle ABC$.

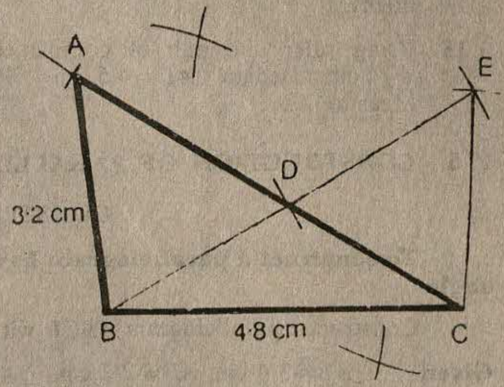
Const. : (1) Draw a triangle *BCE* having its side $BC=4{\cdot}8$ cm, $CE=3{\cdot}2$ cm and $BE=2\times 2{\cdot}8$ cm *i.e*, 5·6 cm.

(2) Bisect segment *BE* at *D*.

(3) Join *CD* and produce it to *A* making $DA=CD$.

(4) Join *AB*.

Then *ABC* is the required triangle.



EXERCISE—17 (b)

(Section—A)

1. Construct an equilateral triangle whose height is 2·8 cm.
2. Construct an isosceles triangle with base=3·9 cm and altitude=3·2 cm.
3. Construct a triangle *ABC* having given the altitude $AD=3{\cdot}2$ cm. $\angle B=45^\circ$ and $\angle C=60^\circ$.
4. Using ruler and compasses only, construct a $\triangle ABC$ such that $BC=6$ cm, $\angle A=60^\circ$ and the length of the perpendicular from the vertex *A* to the base *BC* is 2·5 cm.
5. Using ruler and compasses only, construct a triangle *ABC*, having given $\angle ABC=60^\circ$, $\angle ACB=45^\circ$ and the perpendicular from *A* to *BC* equal to 2·5 cm. Measure *AB*.

(Section—B)

6. Construct an acute-angled triangle *PQR* in which the base $QR=4$ cm, $PQ=3{\cdot}5$ cm and altitude $PS=3{\cdot}2$ cm.
7. Construct a $\triangle ABC$, having given $b+c=5{\cdot}6$ cm, $a=4{\cdot}5$ cm, and $\angle B=45^\circ$.
8. Using ruler and compasses only, construct triangle *ABC*, given base $BC=7$ cm, $\angle ABC=60^\circ$ and $AB+AC=12$ cm.
9. Construct a $\triangle ABC$, having given $a=6$ cm, $c-b=2$ cm, and $\angle B=30^\circ$.
10. Using ruler and compasses only, construct a right-angled triangle in which the hypotenuse is 5 cm long and the difference of the lengths of the other two sides is 1 cm. Measure the lengths of the other two sides.
11. Construct a $\triangle PQR$ in which $QR=4$ cm, $PQ=3{\cdot}5$ cm and the median $PS=3$ cm.
12. Using ruler and compasses only, construct a right triangle *ABC* such that $\angle BAC=60^\circ$, hypotenuse $BC=6$ cm and median $AD=4$ cm.

(Section—C)

13. The perimeter of a triangle is 9 cm and the angles at the base are 45° and 60°. Construct the triangle and measure its sides.
14. Using ruler and compasses only, costruct a $\triangle$ *ABC* from the following data :
$AB+BC+CA=12$ cm, $\angle B=45^\circ$, and $\angle C=60^\circ$. Measure *BC*.

15. Using ruler and compasses only, construct an isosceles triangle of height 2 cm and perimeter 8 cm. Measure the base of the triangle. State your steps of construction briefly.

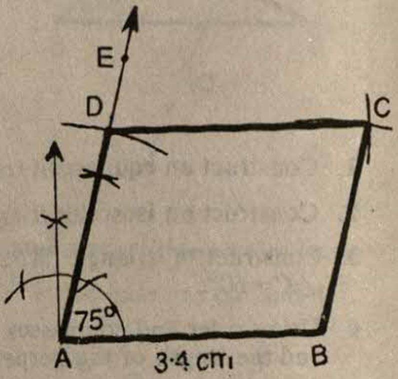
16. Using ruler and pair of compasses only, construct a triangle *ABC*, having given $a=4$ cm, median $AD=4{\cdot}5$ cm, length of perpendicular from *B* to $AC=3$ cm. Measure *AB*.

17 5. CONSTRUCTION OF PARALLELOGRAMS

CONSTRUCTION 18 (*a*)

To construct a parallelogram having given two adjacent sides and the included angle.

Construct a parallelogram *ABCD* with $AB=3{\cdot}4$ cm, $AD=2{\cdot}7$ cm, and $\angle A=75°$.



Given : $AB=3{\cdot}4$ cm, $AD=2{\cdot}7$ cm, $\angle A=75°$.

Required : To construct the parallelogram *ABCD*.

Const. : (1) Draw segment $AB=3{\cdot}4$ cm.

(2) At *A*, make an angle $BAE=75°$.

(3) On ray *AE*, cut off $AD=2{\cdot}7$ cm.

(4) With centre *B* and radius 2·7 cm, draw an arc.

(5) With centre *D* and radius 3·4 cm, draw another arc to cut the former arc at *C*.

(6) Join *BC* and *DC*.

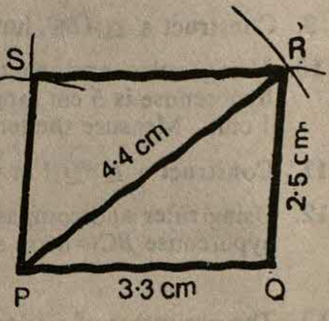
Then *ABCD* is the required parallelogram.

CONSTRUCTION 18 (*b*)

To construct a parallelogram having given two adjacent sides and a diagonal.

Construct a parallelogram *PQRS* in which $PQ=3{\cdot}3$ cm, $QR=2{\cdot}5$ cm, and $PR=4{\cdot}4$ cm.

As the three sides *PQ*, *QR* and *PR* are given, the $\triangle PQR$ can be drawn first. The other two sides *RS* and *SP* are also known, being the opposite sides of a parallelogram. Then $\triangle PRS$ can be drawn and the parallelogram completed.



Given : $PQ=3{\cdot}3$ cm, $QR=2{\cdot}5$ cm and $PR=4{\cdot}4$ cm.

Required : To construct the parallelogram *PQRS*.

Const. : (1) Draw segment $PQ=3{\cdot}3$ cm.

(2) With centre *P* and radius 4·4 cm, draw an arc.

(3) With centre *Q* and radius 2·5 cm, draw another arc to cut the above arc at *R*.

(4) Join *QR* and *PR*.

(5) With centres *P* and *R* and radii equal to 2·5 cm and 3·3 cm respectively, draw arcs intersecting at *S*.

(6) Join *PS* and *SR*.

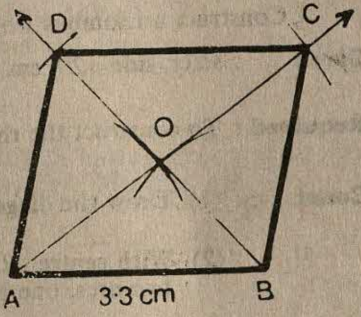
Then *PQRS* is the required parallelogram.

CONSTRUCTION 18 (*c*)

To construct a parallelogram having given two diagonals and one side.

Construct a parallelogram *ABCD* whose diagonals *AC* and *BD* are 5 cm and 4 cm respectively and side $AB=3{\cdot}3$ cm.

Since the diagonals of a parallelogram bisect each other, say at *O*, the sides of the $\triangle AOB$ are known. $AO=\frac{1}{2}AC$, $BO=\frac{1}{2}BD$ and *AB* is given. The other two vertices of the parallelogram are the extremities of the diagonals which can be obtained by producing *AO* and *BO*.



Given : $AB=3{\cdot}3$ cm, and diagonals $AC=5$ cm, $BD=4$ cm.

Required : To construct the parallelogram *ABCD*.

Const. : (1) Draw segment $AB=3{\cdot}3$ cm.

(2) With centre *A* and radius equal to half the length of the diagonal *AC* *i.e.*, 2·5 cm, draw an arc.

(3) With centre *B* and radius equal to $\frac{1}{2}BD$ *i.e.*, 2 cm, draw another arc to cut the former arc at *O*.

(4) Draw ray *AO* and on it cut off $OC=AO=2{\cdot}5$ cm.

(5) Draw ray *BO* and on it cut off $OD=BO=2$ cm.

(6) Join *BC*, *CD* and *DA*.

Then *ABCD* is the required parallelogram.

EXERCISE—17 (i)

(Section—A)

1. Construct a parallelogram *ABCD* in which
   - (*a*) $AB=3{\cdot}4$ cm, $BC=2$ cm and $\angle A=75^\circ$
   - (*b*) $AB=4{\cdot}2$ cm, $AD=3{\cdot}4$ cm and $AC=2$ cm.
2. Using ruler and compasses only, construct a parallelogram *ABCD*, given $AB=6$ cm, $BC=5$ cm and $\angle ABC=60^\circ$.
3. Construct a parallelogram *ABCD* so that $AB=4{\cdot}5$ cm, $BC=3{\cdot}7$ cm and height$=2{\cdot}5$ cm.
4. Construct a parallelogram *ABCD* with the following data :
   - (*a*) $AC=4$ cm, $BD=3$ cm, and $AB=2{\cdot}7$ cm.
   - (*b*) $AC=4{\cdot}4$ cm, $BD=3{\cdot}2$ cm, and angle between them$=45^\circ$.

(Section—B)

5. Using ruler and compasses only, construct a parallelogram *ABCD* given that $AB=4$ cm, $AC=10$ cm, $BD=6$ cm and *AC* and *BD* are the diagonals. Measure *BC*.
6. Use ruler and compasses only to construct a parallelogram with diagonals 6 cm and 8 cm in length having given the acute angle between them is 60°. Measure one of the longer sides.
7. Without using set square or protractor draw a parallelogram whose diagonals are 4 cm and 6 cm long and contain an angle of 75°. Measure and write down the length of one of the shorter sides of the parallelogram.

8. Using ruler and compasses only, draw a parallelogram whose diagonals are 5 cm and 7 cm long and which cut each other, so that one pair of vertically opposite angles are each equal to 75°. Measure the length of the shorter side. State your steps of construction briefly.

17.6. CONSTRUCTION OF RHOMBUSES

CONSTRUCTION 19 (*a*)

To construct a rhombus having given one side and one diagonal.

Construct a rhombus whose each side=2·8 cm and one diagonal=2·5 cm.

Given : Each side=2·8 cm and one diagonal=2·5 cm.

Required : To construct the rhombus.

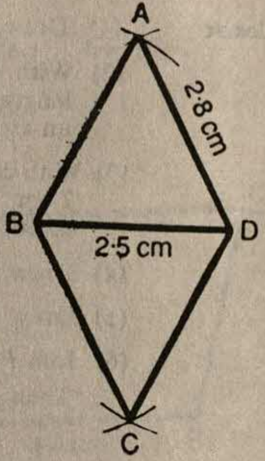
Const. : (1) Draw the diagonal $BD = 2{\cdot}5$ cm.

(2) With centre B and radius equal to 2·8 cm, draw two arcs, one on each side of BD.

(3) With centre D and the same radius, draw two arcs to cut the former arcs at A and C.

(4) Join AB, BC, CD and DA.

Then $ABCD$ is the required rhombus.



CONSTRUCTION 19 (*b*)

To construct a rhombus having given both diagonals.

Construct a rhombus whose diagonals are 3·5 cm and 2·4 cm.

Given : Diagonals 3·5 cm and 2·4 cm in length.

Required : To construct the rhombus.

Const. : (1) Draw diagonal $AC = 3{\cdot}5$ cm.

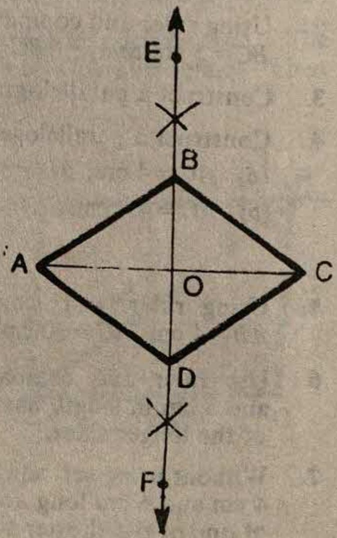
(2) Draw line EOF, the right-bisector of AC.

(3) On ray OE, cut off $OB = \frac{1}{2}$ the other diagonal *i.e.*, $\frac{1}{2} \times 2{\cdot}4$ cm $= 1{\cdot}2$ cm.

(4) On ray OF, cut off $OD = \frac{1}{2} \times 2{\cdot}4$ cm $= 1{\cdot}2$ cm.

(5) Join AB, BC, CD and DA.

Then $ABCD$ is the required rhombus



EXERCISE—17 (j)

(Section—A)

1 Draw a rhombus, having given
   - (*a*) one side=5·6 cm, and one angle=30°
   - (*b*) one side=2·4 cm, and one diagonal=3 cm
   - (*c*) diagonals 3·7 cm and 3 cm.

2. Construct a rhombus *ABCD* of side 4 cm such that $\angle ABC = 60°$.
3. Construct a rhombus *PQRS* in which *PR*=7·6 cm and *QS*=5·2 cm.
4. Construct a rhombus *PQRS* in which diagonal *PR*=30 mm and diagonal *QS*=72 mm. Measure its side.

(Section—B)

5. Using ruler and compasses only, construct a rhombus *ABCD* having given $\angle A = 120°$, diagonal *AC*=4·8 cm. Write the steps of construction briefly. Measure *AB*.

17.7. CONSTRUCTION OF SQUARES

CONSTRUCTION 20 (*a*)

To construct a square having given one side.

Construct a square on a side of 2·6 cm.

Steps of Construction :

(1) Draw segment *AB*=2·6 cm.

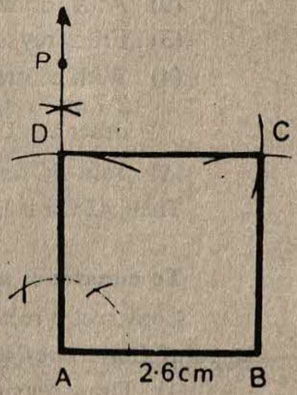
(2) Draw ray $AP \perp AB$.

(3) From ray *AP*, cut off *AD*=2·6 cm.

(4) With centre *B* and radius 2·6 cm draw an arc.

(5) With centre *D* and the same radius draw another arc cutting the former arc at *C*.

(6) Join *BC* and *CD*.
Then *ABCD* is the required square.



CONSTRUCTION 20 (*b*)

To construct a square having given a diagonal.

Construct a square on a diagonal of 3·4 cm.

Steps of Construction :

(1) Draw a diagonal *AC*=3·4 cm.

(2) At *A*, draw ray $AM \perp AC$.

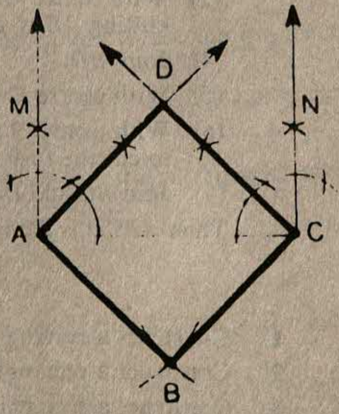
(3) At *C*, draw ray $CN \perp CA$.

(4) Draw the bisectors of the angles *CAM* and *ACN*. (Make 45°).

Let them intersect at *D*.

(5) With centres *A* and *C* and radius equal to *AD*, draw arcs cutting each other at *B*.

(6) Join *AB* and *BC*.
Then *ABCD* is the required square.



EXERCISE—17 (k)

(Section—A)

1. Construct a square whose one side = 2·7 cm.
2. Construct a square whose one diagonal = 4·4 cm.
3. Construct a square whose perimeter is 9·6 cm.
4. Using ruler and compasses only construct a square whose diagonals measure 5 cm each. Measure the side of the square to the nearest millimetre.

(Section—B)

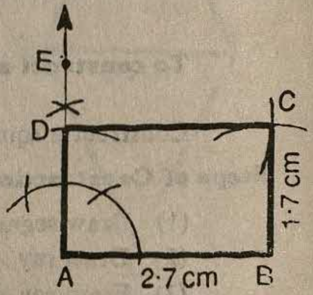
5. Construct a square *ABCD* of side 3 cm and construct another square with *AC* as one side.
6. Construct a square when the sum of one diagonal and two sides is equal to 10·8 cm.

17.8. CONSTRUCTION OF RECTANGLES

CONSTRUCTION 21 (*a*)

To construct a rectangle having given adjacent sides.

Construct a rectangle whose adjacent sides are 2·7 cm and 1·7 cm.



Steps of Construction :

(1) Draw segment $AB = 2{\cdot}7$ cm.

(2) At *A*, draw ray $AE \perp AB$.

(3) From ray *AE*, cut off $AD = 1{\cdot}7$ cm.

(4) With centres *B* and *D* and radii equal to 1·7 cm and 2·7 cm respectively, draw two arcs intersecting each other at *C*.

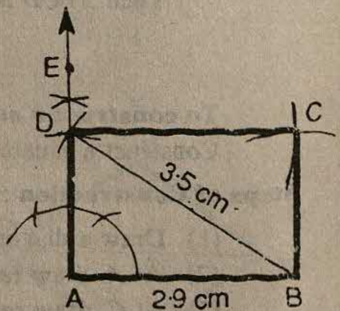
(5) Join *BC* and *CD*.

Then *ABCD* is the required rectangle.

CONSTRUCTION 21 (*b*)

To construct a rectangle having given one side and one diagonal.

Construct a rectangle when a diagonal = 3·5 cm and one side = 2·9 cm.



Steps of Construction :

(1) Draw segment $AB = 2{\cdot}9$ cm.

(2) At *A*, draw ray $AE \perp AB$.

(3) With centre *B* and radius = 3·5 cm, draw an arc cutting *AE* at *D*.

(4) Join *BD*.

(5) With centre *D* and radius = 2·9 cm, draw an arc.

(6) With centre *B* and radius = *AD*, draw another arc to cut the former arc at *C*.

(7) Join *BC* and *DC*.

Then *ABCD* is the required rectangle.

EXERCISE – 17 (l)

(Section—A)

1. Construct a rectangle whose adjacent sides are 2·8 cm and 3·9 cm.
2. Construct a rectangle in which the diagonal = 4·7 cm, and one side = 3·3 cm.
3. Construct a rectangle whose area is 5 cm^2.

(Section—B)

4. Construct a rectangle in which one diagonal is 3·7 cm and makes an angle of 30° with the other diagonal.

5. Using ruler and compasses only, construct a rectangle each of whose diagonals measures 6 cm and the diagonals intersect at an angle of 45°.

17.9. CONSTRUCTION OF TRAPEZIUM

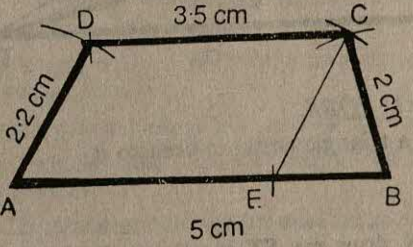
CONSTRUCTION 22

Construct a trapezium $ABCD$ in which $AB=5$ cm, $BC=2$ cm, $CD=3{\cdot}5$ cm, $DA=2{\cdot}2$ cm and AB is parallel to CD.

Let us make a rough sketch of the trapezium $ABCD$ and mark the data on it. If we draw CE parallel to DA, we get a parallelogram $AECD$ and $\triangle BCE$. $\triangle BCE$ can be drawn as its three sides are known.

$EB=AB-AE=AB-DC=5\text{ cm}-3{\cdot}5\text{ cm}=1{\cdot}5$ cm, $CE=DA=2{\cdot}2$ cm.

Then the $\parallel^{gm}$ $AECD$ also can be drawn and the figure completed.



Given : $AB=5$ cm, $BC=2$ cm, $CD=3{\cdot}5$ cm, $DA=2{\cdot}2$ cm and $AB \parallel CD$.

Required : To construct the trapezium $ABCD$,

Steps of Construction :

(1) Draw a line segment $AB=5$ cm.

(2) From BA, cut off $BE=1{\cdot}5$ cm.

(3) With centre B and radius equal to 2 cm, draw an arc.

(4) With centre E and radius equal to 2·2 cm, draw another arc intersecting the former arc at C. **[$\triangle BCE$ is drawn]**

(5) Join BC and EC.

(6) With centre A radius equal to 2·2 cm, draw an arc.

(7) With centre C and radius equal to 3·5 cm, draw another arc to cut the former arc at D. **[$\parallel^{gm}$ $AECD$ is completed]**

(8) Join CD and AD.

Then $ABCD$ is the required trapezium.

When the four sides of a trapezium are given, always divide it into a triangle and a parallelogram and construct the triangle first.

EXERCISE—17 (m)

(Section—A)

1. Construct a trapezium $ABCD$ with $AB \parallel DC$ and $AB=4{\cdot}5$ cm, $BC=1{\cdot}8$ cm, $CD=2$ cm, $DA=1{\cdot}4$ cm.

2. Draw a trapezium $ABCD$ in which $AB \parallel DC$. $AB=8$ cm, $BC=4$ cm, $CD=DA=5$ cm.

(Section—B)

3. Draw a trapezium $PQRS$ in which $PQ \parallel SR$ and $PQ=4{\cdot}5$ cm, $RS=3$ cm, $SP=3$ cm and $\angle QPS=105°$.
4. Construct a trapezium $ABCD$ with $AB \parallel DC$ and $AB=3{\cdot}8$ cm, $CD=6$ cm, $DA=3{\cdot}5$ cm and $\angle A=120°$.

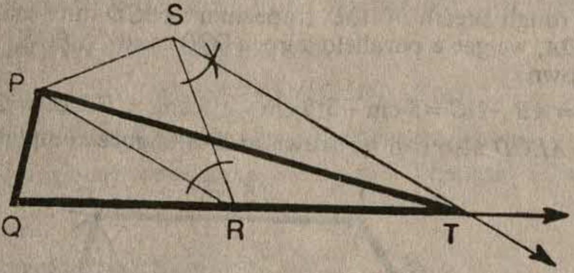
(Section—C)

5. Construct an isosceles trapezium in which the parallel sides are equal to 4·5 cm and 2·5 cm respectively and non-parallel sides are each equal to 2 cm.

17.10. CONSTRUCTION OF EQUIVALENT AREA

CONSTRUCTION 23

To construct a triangle equal in area to a given quadrilateral.



Given : A quadrilateral $PQRS$.

Required : To construct a triangle equal in area to it.

Steps of Construction :

(1) Join PR.

(2) Through S, draw ray ST parallel to PR intersecting ray QR at T.

(3) Join PT.

Then $\triangle PQT$ is the required triangle.

Proof : $\triangle PRT$ and $\triangle PRS$ stand on the same base PR and lie between the same parallels PR and ST.

$$\therefore \quad ar(\triangle PRT)=ar(\triangle PRS)$$

$$\begin{aligned}\text{Then } ar\ (\text{quad. } PQRS) &= ar(\triangle PQR)+ar(\triangle PRS)\\ &= ar\,(\triangle PQR)+ar(\triangle PRT)\\ &= ar(\triangle PQT)\end{aligned}$$

Whenever an equivalent triangle is to be constructed on any side of quadrilateral, take the proper diagonal. *Do not produce that side.*

EXERCISE—17 (n)

1. Construct a quadrilateral $ABCD$ in which $AB=3{\cdot}5$ cm, $AC=4{\cdot}5$ cm $AD=5$ cm and $\angle ABC=\angle ACD=90°$.

   Construct a triangle equal in area to the quadrilateral.
2. Draw a quadrilateral $ABCD$ from the following data :

   $\angle A=90°$, $AB=7{\cdot}7$ cm, $AD=3{\cdot}6$ cm, $BC=6{\cdot}8$ cm and $DC=5{\cdot}1$ cm.

   Reduce the quadrilateral into a triangle of equal area, stating the steps of construction without proof.
3. Construct an isosceles trapezium in which the parallel sides are equal to 7 cm and 5 cm and the non-parallel sides are each equal to 3 cm.

   On a non-parallel side as base, construct a triangle equal in area to this trapezium,

REVIEW EXERCISE—XVII

(Section—A)

1. Fill in the blanks making each of the following a true statement :

   (*a*) Every construction problem depends on one or more.........constructions.

   (*b*) There is no unique triangle, if only three........ are given.

   (*c*) It is always possible to construct exactly one line through any two.........points.

   (*d*) ....independent data, in all, are required to construct a quadrilateral.

   (*e*) To bisect a given line segment, we construct the................. of the given segment.

   (*f*) If two circles of radius r_1 and r_2 have their centres at a distance of d, they intersect only if ....

(Section—B)

2. Take a line segment AB of length 5 cm. Mark a point C on it such that $AC=2$ cm. Through C, draw a line perpendicular to AB.
3. Draw any triangle ABC. Construct $AD \perp BC$.
4. Draw any triangle PQR. Through vertex P, construct a line parallel to the opposite side.
5. Draw a line segment AB of length 5 cm. With ruler and compasses, divide AB in the ratio 1 : 2. Measure the longer part.
6. Construct an isosceles triangle whose base is 3·6 cm and the vertical angle is 30°.
7. Construct a parallelogram whose sides are of length 4·5 cm and 3·9 cm and one of whose diagonals is of length 6 cm.
8. Construct a trapezium $ABCD$, when one of the parallel sides $AB=5$ cm, height$=2{\cdot}5$ cm, $BC=3$ cm and $AD=3{\cdot}3$ cm.
9. Construct a square whose one diagonal is 5·4 cm.
10. Draw a rhombus with a diagonal equal to 4·5 cm and each side equal to 3·7 cm.
11. Construct a parallelogram $ABCD$, if $AC=7{\cdot}2$ cm and $BD=5{\cdot}4$ cm and the angle included between the diagonals is 30°.

(Section—C)

12. Construct a triangle with base of length 5 cm and the sum of the other two sides 7·8 cm and one base angle of 60°.
13. Construct a $\triangle ABC$ in which $BC=4$ cm, $AB=5$ cm and median $BE=3{\cdot}5$ cm.
14. Construct an isosceles triangle when base$=4$ cm and the sum of one side and the altitude$=6{\cdot}4$ cm.
15. Construct a $\triangle ABC$ when $\angle B=60°$, $\angle C=45°$ and $AB+BC=7{\cdot}5$ cm.
16. Construct an isosceles triangle whose perimeter is 11 cm and altitude 2·5 cm.
17. Construct a triangle whose perimeter is 10 cm and base angles are 30° and 60°.
18. Construct a triangle whose perimeter is 13·5 cm and ratio of the three sides is 2 : 3 : 4.
19. Construct a square $ABCD$ in which $AB+BC+AC=9{\cdot}5$ cm.
20. Construct a quadrilateral $ABCD$, having $AB=5$ cm, $BC=7{\cdot}5$ cm and the angles A, B and D equal respectively to 125°, 70° and 95°. Construct a triangle, having BC as one of its sides and equal in area to the quadrilateral. Measure the other two sides of the triangle.

□□

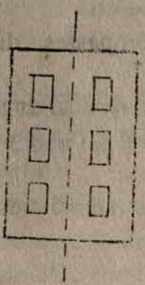
18

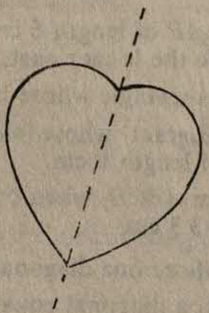
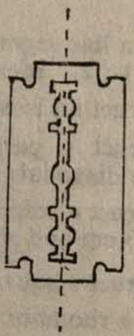
LINEAR SYMMETRY

18.1. LINE OF SYMMETRY

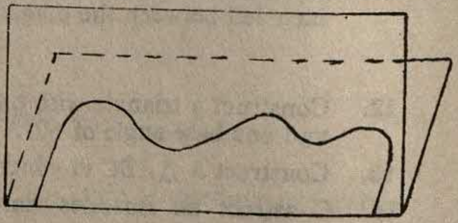
Two geometric figures are congruent if they are exactly alike in all respects, that is they have exactly the same shape and size. Sometimes we find that parts of the same figure are exactly alike *i.e.*, have same shape and size.

Observe the following figures. The parts of the figures on one side of the dotted line match the parts on the other side.



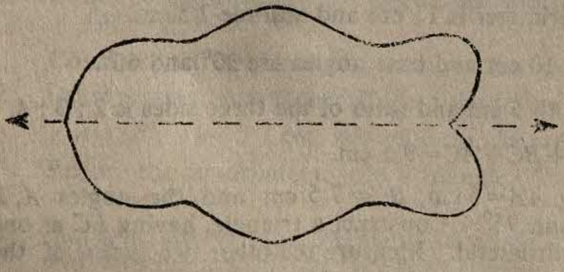
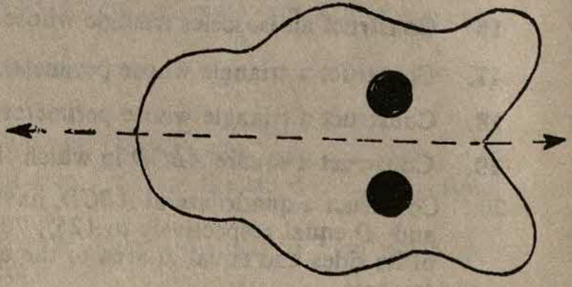



We say that the figures have *symmetry* about the broken line. The broken line is called the *line of symmetry*.



Fold a piece of paper and cut a curve along the crease. Open the paper. The figure on thus got is symmetrical about the crease. The crease is the line of symmetry.

Again fold a paper. Put an ink drop between the folds and press. Carefully open out.

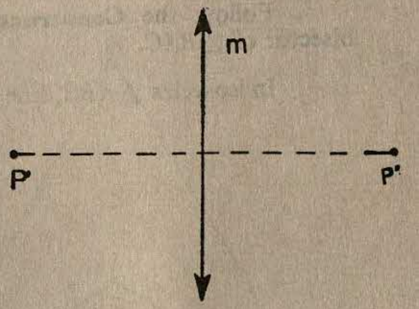



Once again you find figures symmetrical about the crease. When we fold a symmetric figure along the line of symmetry the two parts exactly coincide.

Take a strip of plane mirror and place it vertical on the line of symmetry in any of the above figures. You will see that one part of the figure is reflected on the other part. Thus, we say that parts on either side of the line of symmetry are **mirror image** of each other.

When you look into a mirror, you see your image behind the mirror. This image is due to **reflection** in the mirror.

You know that the image is as far behind the mirror as the object is infront.



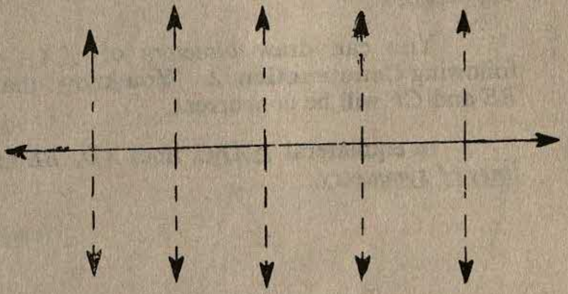
If P is the image of P' upon reflection in mirror m, the line m is the perpendicular bisector of the line segment PP'.

The line m is called the **mirror line** or the **reflection line** or the **mediator**.

The mirror line is also called **line of symmetry** or **axis of symmetry.**

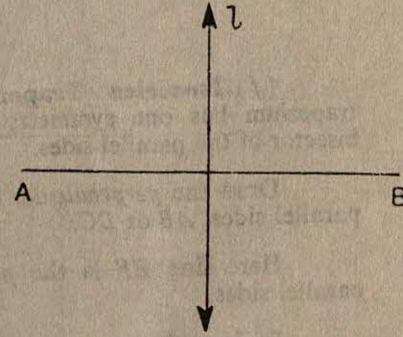
Remember that an axis of symmetry of a figure divides in into two *congruent figures.*

Some figures have more than one line of symmetry. Let us now consider the following figures :



(*a*) **Lines.** A line has an infinite number of symmetry lines, all perpendicular to it.

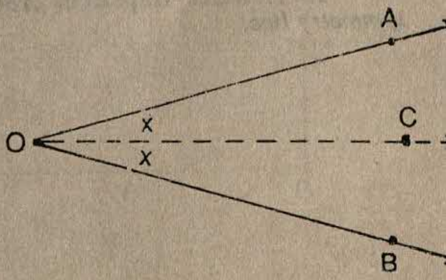
The line itself coincides with its image.



(*b*) **Segments.** A line segment has two symmetry lines, its perpendicular bisector and the line segment itself.

The perpendicular bisector of line segment AB can be drawn by following **Construction 1.**

Here line l is the *perpendicular bisector* of line segment AB. So line l is *line of symmetry* of line segment AB.



(*c*) **Angles**. An angle has one symmetry line, its angular bisector.

The bisector of $\angle AOB$ can be drawn by following the **Construction 2.**

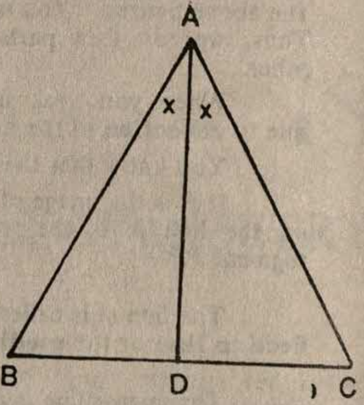
Here line OC is the *bisector of* $\angle AOB$.

So line OC is the *line of symmetry* of $\angle AOB$.

(*d*) **Isosceles Triangles.** An isosceles triangle has one symmetry line, the angular bisector of the vertex which is also the perpendicular bisector of the base.

Follow the **Construction** 2 to draw line *AD*, the bisector of $\angle BAC$.

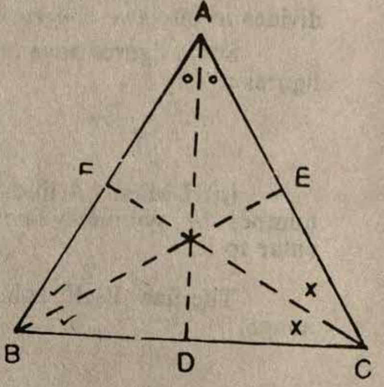
In isosceles $\triangle ABC$, line *AD* is the *line of symmetry.*



(*e*) **Equilateral Triangles.** An equilateral triangle has three symmetry lines, the angular bisector of each of the vertices.

You can draw *bisectors* of $\angle A$, $\angle B$ and $\angle C$ by following **Construction 2.** You know that bisectors *AD*, *BE* and *CF* will be concurrent.

In equilateral $\triangle ABC$, lines *AD*, *BE* and *CF* are three *lines of symmetry.*

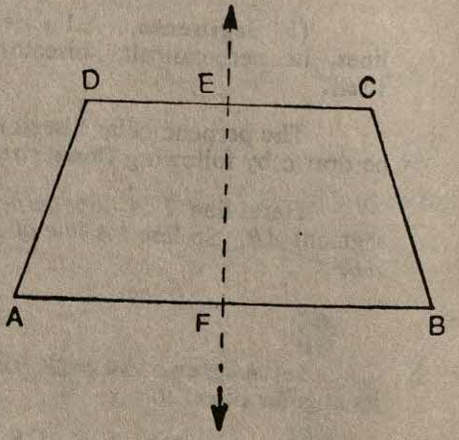


(*f*) **Isosceles Trapeziums.** An isosceles trapezium has one symmetry line, the perpendicular bisector of the parallel sides.

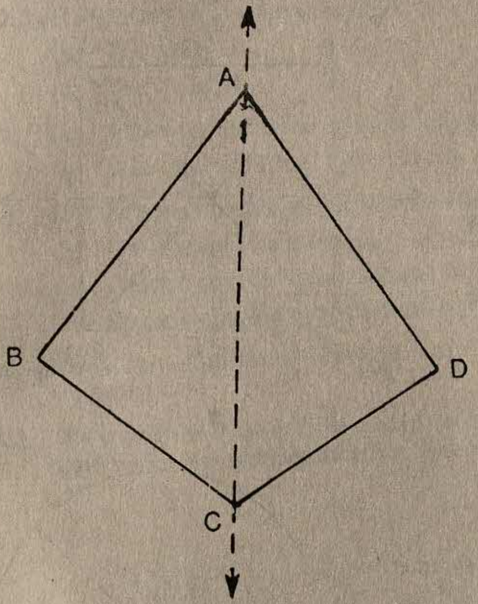
Draw the *perpendicular bisector* of one of the parallel sides, *AB* or *DC.*

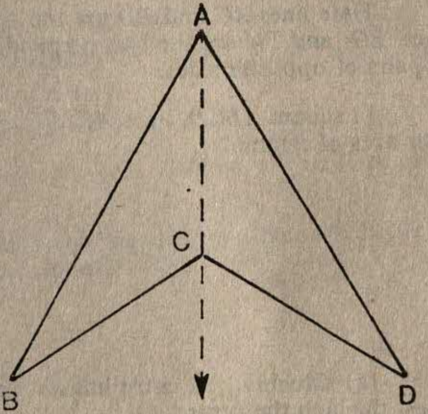
Here line *EF* is the *perpendicular bisector* of parallel sides.

In isosceles trapezium *ABCD*, line *EF* is the *symmetry line.*



(*g*) **Kites and Arrow Heads.** One of their diagonals is a symmetry line.





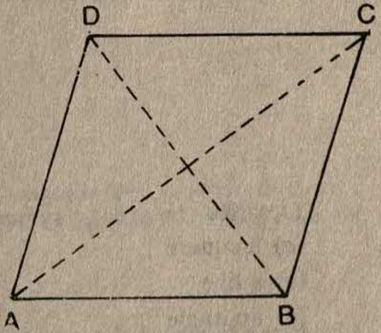
You can draw a diagonal *AC* by joining the opposite vertices *A* and *C*.

Line *AC* is the *line of symmetry* of kite *ABCD*.

(*h*) **Rhombuses.** A rhombus has two symmetry lines, its diagonals.

Here lines *AC* and *BD* are the two diagonals of rhombus *ABCD*.

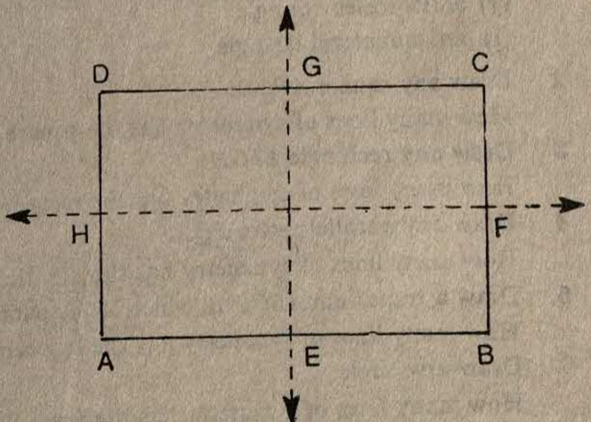
In rhombus *ABCD*, lines *AC* and *BD* are two lines of symmetry.

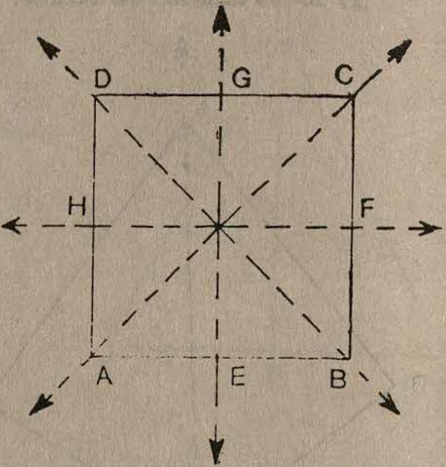


(*i*) **Rectangles.** A rectangle has two symmetry lines, the perpendicular bisector of each pair of opposite sides.

Here line *EG* is the perpendicular bisector of opposite sides *AB* and *DC*; line *FH* is the perpendicular bisector of opposite sides *BC* and *AD*.

So, in rectangle *ABCD*, lines *EG* and *FH* are two lines of symmetry.

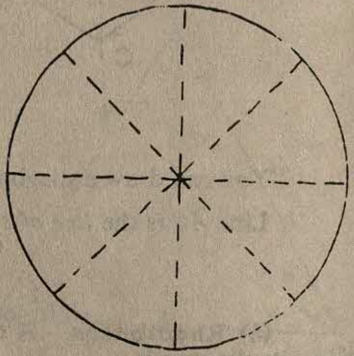




(*j*) **Squares**. A square has four symmetry lines, the two diagonals and the perpendicular bisector of each pair of opposite sides.

Here lines *AC* and *BD* are the two diagonals; lines *EG* and *FH* are the two perpendicular bisectors of pairs of opposite sides.

In square *ABCD*, lines *AC*, *BD*, *EG* and *FH* are four axes of symmetry.



(*k*) **Circles.** A circle has an infinite set of symmetry lines, through the centre.

Note that an infinite number of diameters can be drawn in a given circle.

Any diameter of a circle is its axis of symmetry.

EXERCISE—18 (a)

(Section—A)

1. Describe the axes of symmetry of
   - (*a*) a square
   - (*b*) a circle
   - (*c*) a line
   - (*d*) a rectangle
   - (*e*) an angle
   - (*f*) a rhombus
   - (*g*) an isosceles triangle
   - (*h*) an isosceles trapezium
   - (*i*) an equilateral triangle
   - (*j*) a line segment.
2. Draw any square *ABCD*.
   How many lines of symmetry has the square ?
3. Draw any rectangle *EFGH*.
   How many lines of symmetry has the rectangle ?
4. Draw any parallelogram *PQRS*.
   How many lines of symmetry has the parallelogram ?
5. Draw a trapezium *ABCD* in which *AB* is parallel to *DC*.
   How many lines of symmetry has the trapezium ?
6. Draw any circle.
   How many lines of symmetry has the circle ?

(Section—B)

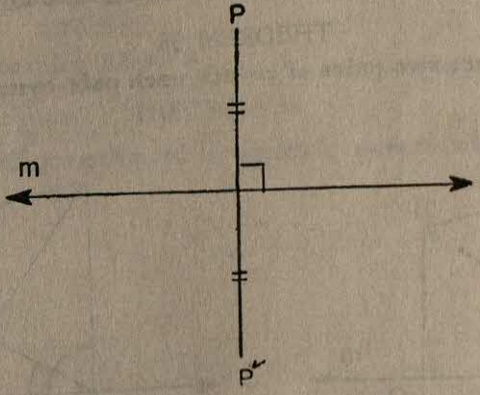
7. How many lines of symmetry have each of the following ?
   (*a*) The letter *A* (*b*) The letter *H*
   (*c*) The letter *B* (*d*) The letter *W*

8. (*a*) What quadrilateral have exactly one axis of symmetry which is a diagonal ?
   (*b*) What quadrilateral have exactly one axis of symmetry which is not a diagonal ?

9. (*a*) Draw a specific quadrilateral which has no axes of symmetry.
   (*b*) Draw a specific quadrilateral with one axis of symmetry.
   (*c*) Draw a specific quadrilateral with two axes of symmetry.
   (*d*) Draw a specific quadrilateral with four axes of symmetry.

10. Draw an equilateral triangle each of whose sides is 4 cm. Draw all its lines of symmetry.

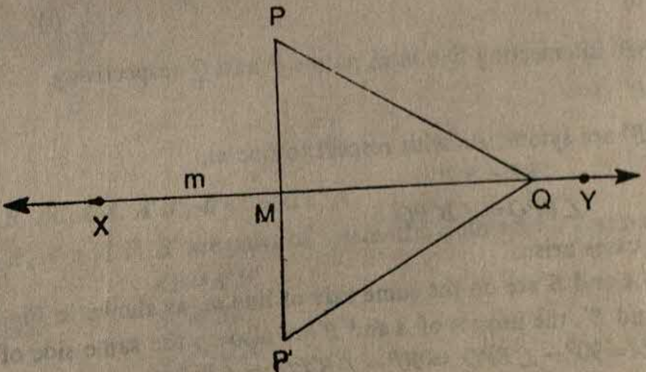
18.2. **Two points P and P′ are said to be symmetric with respect to a line m, if the line m is the perpendicular bisector of the line segment PP′.**



Also, each of the two points *P* and *P′*, is said to be **symmetric** to the other with respect to the line *m*, and the line *m* is called the **axis of symmetry** or **the line of symmetry** of the line segment joining these two points.

THEOREM 37

A pair of points symmetric with respect to a line, are equidistant from any point on the line of symmetry.



Let m be the line of symmetry and a pair of points P and P', symmetric with respect to line m.

Proof : Join PP' intersecting line m in M.

$PM=P'M$ and $PM \perp m$ [By definition]

Let Q be any point of line m.

Join PQ and $P'Q$.

Now in ΔPMQ and $\Delta P'MQ$

$PM=P'M$ [Proved above]

$QM=QM$

Incl. $\angle PMQ=$Incl. $\angle P'MQ$ [Each=90°]

$\therefore \quad \Delta PMQ \cong \Delta P'MQ$ [*SAS*—Congruency Axiom]

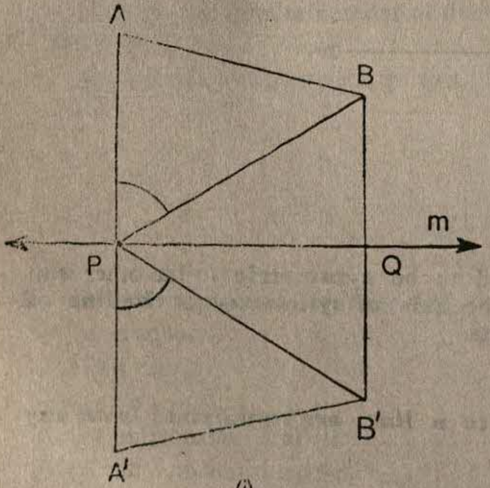
Hence $PQ=P'Q$

i.e., Q is equidistant from symmetric points P and P'.

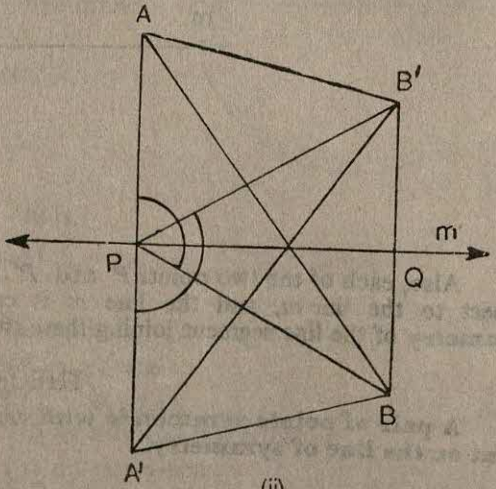
Corollary. If P and P' are a pair of points, symmetric with respect to a line m, and Q is any point of the line m, then PQ and $P'Q$ are equally inclined to the line m.

THEOREM 38

If A, A′ and B, B′ are two pairs of points, each pair symmetric with respect to a line m, then AB=A′B′.



(i)



(ii)

Join AA' and BB' intersecting line m in points P and Q respectively.

Join PB and PB'.

Proof : Since B, B' are symmetric with respect to line m,

$BP=B'P$

and $\angle BPQ=\angle B'PQ$

Now two cases arise.

Case (*i*) : A and B are on the same side of line m, as shown in Fig. (*i*).

Then A' and B', the images of A and B are also on the same side of line m.

$\therefore \quad \angle BPA=90°-\angle BPQ=90°-\angle B'PQ=\angle B'PA'$.

In $\triangle APB$ and $\triangle A'PB'$

$AP = A'P$

$BP = B'P$ [Proved above]

Incl. $\angle BPA =$ Incl. $\angle B'PA'$ [Proved above]

$\therefore$ $\triangle APB \cong \triangle A'PB'$ [*SAS*—Congruency Axiom]

Hence $AB = A'B'$

Case (*ii*) : A and B are on the opposite sides of line m, as shown in Eig. (*ii*).

Then A' and B', the images of A and B are also on the opposite sides of line m.

$\therefore \quad \angle BPA = 90^\circ + \angle BPQ = 90^\circ + \angle B'PQ = \angle B'PA'$.

In $\triangle APB$ and $\triangle A'PB'$

$AP = A'P$

$BP = B'P$ [Proved above]

Incl. $\angle BPA =$ Incl. $\angle B'PA'$ [Proved above]

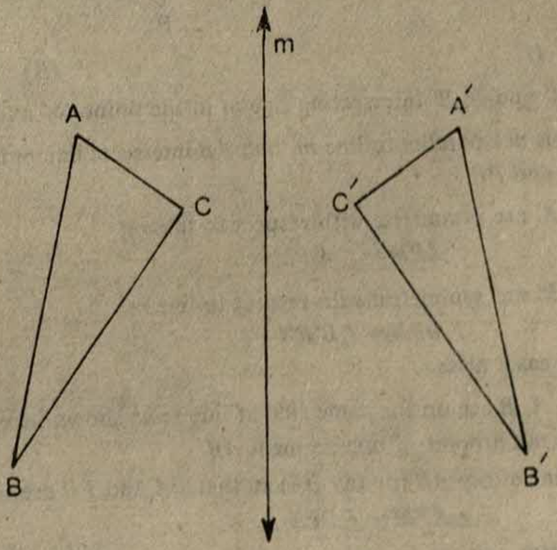
$\therefore$ $\triangle APB \cong \triangle A'PB'$ [*SAS*—Congruency Axiom]

Hence $AB = A'B'$

Thus in both the cases, $AB = A'B'$.

THEOREM 39

If A, A′ ; B, B′, C, C′ are pairs of symmetric points with respect to a line m, then ∠ABC = ∠A′B′C′.



Join AB, BC, CA, $A'B'$, $B'C'$ and $C'A'$.

Proof : Since A, A' and B, B' are pairs of symmetric points with respect to line m.

$AB = A'B'$

Similarly $BC = B'C'$

and $CA = C'A'$

Now in $\triangle ABC$ and $\triangle A'B'C'$

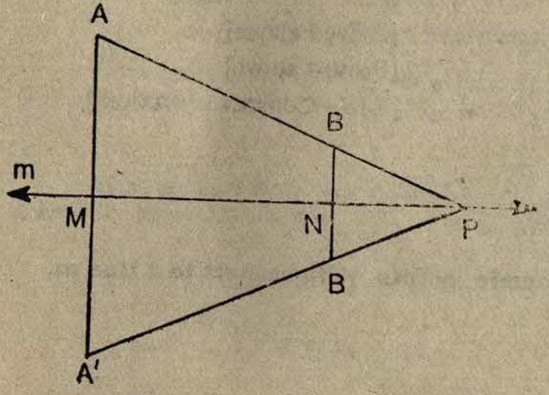
$$AB=A'B'$$
$$BC=B'C'$$
$$CA=C'A'$$

$\therefore$ $\triangle ABC \cong \triangle A'B'C'$ [*SSS*—Congruency Theorem]

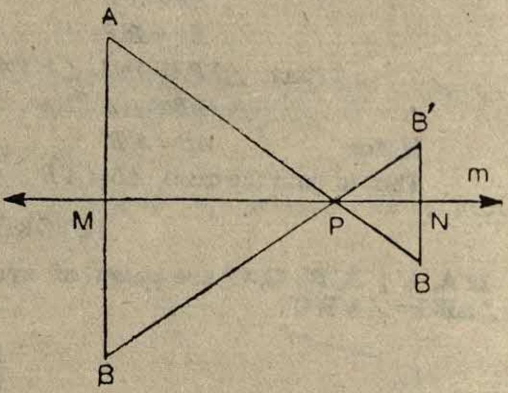
Hence $\angle ABC=\angle A'B'C'$ [Corresponding parts of Congruent Triangles]

THEOREM 40

If A, A′ and B, B′ are pairs of points, each pair symmetric with respect to a line m, and if AB is not parallel to m, then the lines AB and A′B′ intersect line m.



(*i*)



(*ii*)

Join A, A' and B, B' intersecting line m in the points M and N respectively.

Proof : Since AB is not parallel to line m, line AB intersects line m in some point P. Join PA' and PB'.

Since A, A' are symmetric with respect to line m,

$$\angle APM=\angle A'PM \qquad ...(1)$$

Since B, B' are symmetric with respect to line m,

$$\angle BPN=\angle B'PN \qquad ...(2)$$

Now two cases arise.

Case (*i*) : A, B are on the same side of line m as shown in Fig. (*i*).

Then P is not a point of line segment AB.

P is a point of ray AB (or ray BA) so that PA and PB are the same ray.

$$\therefore \quad \angle APM=\angle BPN \qquad ...(3)$$

From (1), (2) and (3), we get

$$\angle A'PM=\angle B'PN$$

Also A', B' lie on the same side of line m.

$\therefore$ PA' and PB' are the same ray.

Hence line $A'B'$ intersects line m at P.

i.e., lines AB and $A'B'$ intersect on line m.

Case (*ii*) : A, B are on opposite sides of line m, as shown in Fig. (*ii*).

Then P is a point of line segment AB.

$\therefore \quad \angle APM = \angle BPN$ [Vertically opposite angles] ...(4)

From (1), (2) and (4), we have

$\angle A'PM = \angle B'PN$ [Vertically opposite angles]

Also A', B' lie on opposite sides of line m.

$\therefore$ $A'P$ and PB' are in the same line.

i.e., line $A'B'$ intersects line m in P.

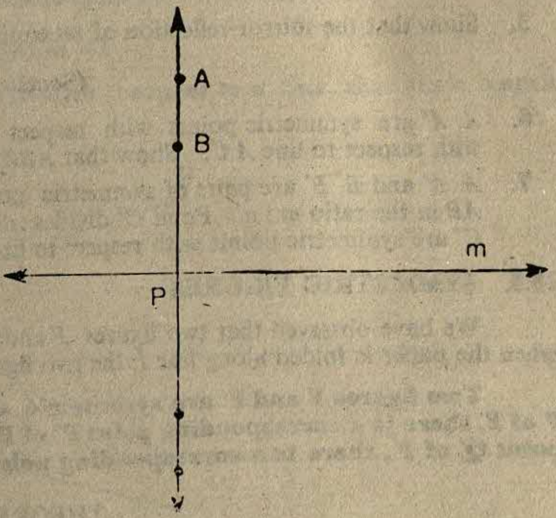
Hence, lines AB and $A'B'$ intersect on line m.

EXERCISE—18 (b)

(Section—A)

1. In the figure, A, A' are symmetric points with respect to line m.

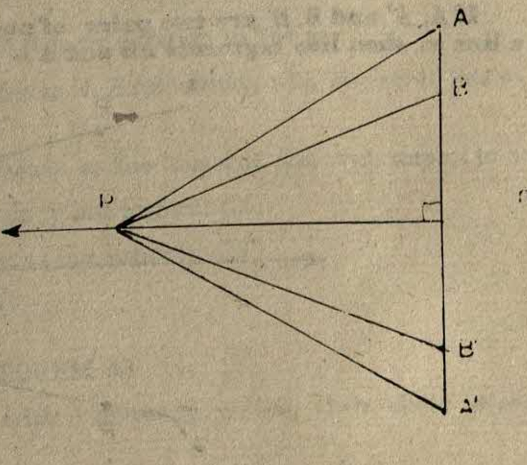
   If $AB = A'B'$, show that C, C' are symmetric with respect to line m.



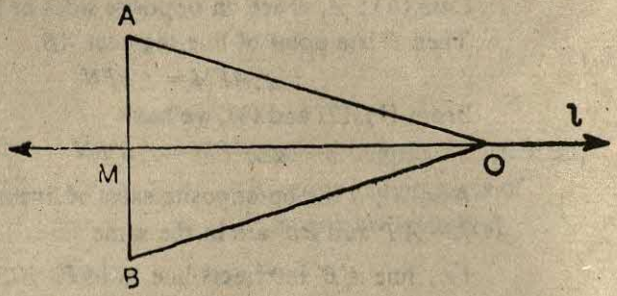
2. In the given figure, A, A' and B, B' are pairs of symmetric points with respect to line m.

   Prove that

   $\angle APB = \angle A'PB'$



3. In the figure, $OA = OB$ and $AM = MB$. Show that line l is the axis of symmetry of line segment AB.



(Section—B)

4. A, A' ; B, B' ; C, C' are pairs of symmetric points with respect to line m. Show that the perimeter of $\triangle ABC$ is equal to the perimeter of $\triangle A'B'C'$.

5. Show that the mirror-reflection of an equilateral triangle is also an equilateral triangle.

(Section—C)

6. A, A' are symmetric points with respect to line BB' and B, B' are symmetric points with respect to line AA'. Show that $ABA'B'$ is a rhombus.

7. A, A' and B, B' are pairs of symmetric points with respect to line m. Point C divides AB in the ratio $m : n$. Point C' divides $A'B'$ in the same ratio $m : n$. Prove that C and C' are symmetric points with respect to line m.

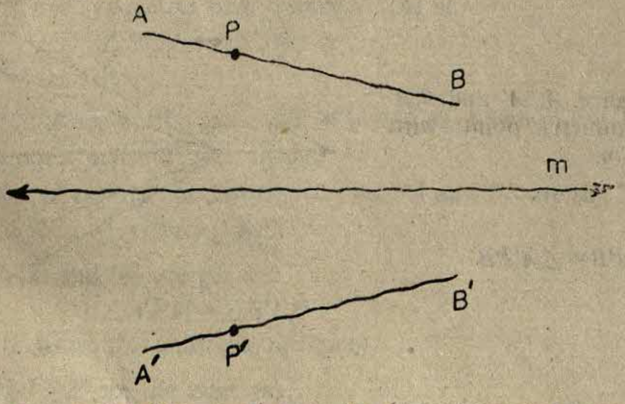
18.3. SYMMETRIC FIGURES

We have observed that two figures F and F' are symmetric with respect to a line l, if when the paper is folded along line l, the two figures coincide with each other.

Two figures F and F′ are symmetric with respect to a line l, if for every point P of F, there is a corresponding point P′ of F′ which is symmetric to P and for every point Q′ of F′, there is a corresponding point Q of F symmetric to Q′.

THEOREM 41

If A, A′ and B, B′ are two pairs of points, each pair symmetric with respect to a line m, then line segments AB and A′B′ are symmetric with respect to line m.



Proof : Let P be any point of line segment AB and P' be the point symmetric to P with respect to line m.

Now A, A' and P, P' are two pairs of points, each pair symmetric with respect to line m.

$\therefore$ $AP = A'P'$ ...(1)

Similarly $PB = P'B'$ and $AB = A'B'$ ...(2)

P is a point of line segment AB.

$\therefore$ $AP + PB = AB$ ...(3)

From (1), (2) and (3), we get

$$A'P' + P'B' = A'B'$$

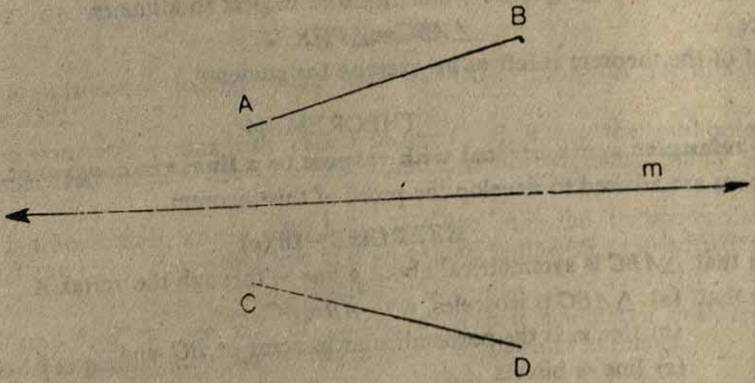
Then P' is a point of line segment $A'B'$.

If Q' is a point of line segment $A'B'$, and Q is symmetric to Q' with respect to line m, then Q is a point of AB.

Hence, line segments AB and $A'B'$ are symmetric with respect to line m.

THEOREM 42

If two line segments are symmetric with respect to a line, then their lengths are equal.



Let line segments AB and CD be symmetric with respect to line m.

Proof : Let A', B' be the points symmetric to A, B respectively with respect to line m.

$\therefore$ $A'B' = AB$

Also, line segment $A'B'$ is symmetric to line segment AB, with respect to line m.

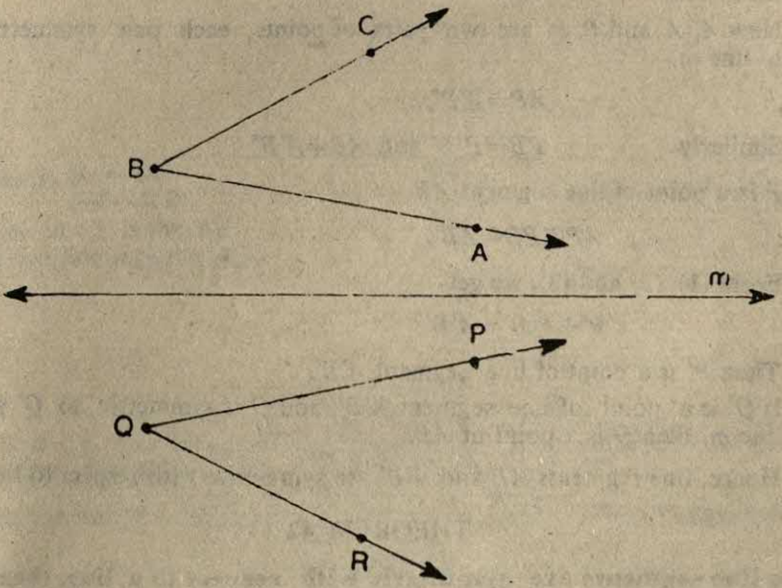
But line segment AB is symmetric to line segment CD.

$\therefore$ line segment CD is the same as line segment $A'B'$.

Hence $AB = CD$

THEOREM 43

If two angles are symmetrical with respect to a line, then their measures are equal.



Here $\angle ABC$ and $\angle PQR$ are symmetric with respect to a line m.

Hence $\angle ABC = \angle PQR$

Proof of the theorem is left as an exercise for students.

THEOREM 44

Two triangles symmetrical with respect to a line, are congruent.

Students are advised to develop the proof of this theorem.

EXERCISE—18 (c)

1. Given that $\triangle ABC$ is symmetrical about a line m through the vertex A.
   Show that (*a*) $\triangle ABC$ is isosceles, with $AB = AC$,
   (*b*) line m is the perpendicular bisector of BC, and
   (*c*) line m bisects $\angle A$.
2. Quadrilateral $ABCD$ is symmetrical about each of its diagonal AC and BD.
   Prove that (*a*) the diagonals bisect each other at right angles, and
   (*b*) the quadrilateral $ABCD$ is a rhombus.
3. Quadrilateral $ABCD$ has two axis of symmetry which are not diagonals of it. Prove that $ABCD$ is a rectangle.
4. Quadrilateral $ABCD$ has four axis of symmetry, two of which are its diagonals. Prove that $ABCD$ is a square.
5. A quadrilateral $ABCD$ has exacty one axis of symmetry, which is not a diagonal. Prove that quadrilateral $ABCD$ is an isosceles trapezium.

REVIEW EXERCISE—XVII

(Section—A)

1. State for each of the following statements whether it is True or False :
   (*a*) Every diameter of a circle is an axis of symmetry for the circle.
   (*b*) The mirror-reflection of an equilateral triangle is also an equilateral triangle.
   (*c*) Any two symmetrical points are equidistant from any point on the line of symmetry.
   (*d*) An isosceles trapezium has no line of symmetry.
   (*e*) A square has four symmetry lines.
   (*f*) Two equal angles are always symmetrical.

2. Fill in the blanks to make the following statements true :
   (*a*) If two line segments are symmetric with respect to a line, their lengths are..........
   (*b*) An equilateral triangle has.........symmetry lines.
   (*c*) Two congruent triangles are........ always symmetrical.
   (*d*) Two triangles symmetrical with respect to a line, are..........
   (*e*) If P is the image of P' upon reflection in mirror m, the line m is.........of the line segment PP'
   (*f*) Our own human body is a.........object.

(Section—B)

3. $\triangle ABC$ is an isosceles triangle such that $AB=AC$. Find the line of symmetry of the triangle ABC.

4. A, A' are symmetric points with respect to line l. B, B' are the mid-points of OA and OA' respectively when O lies on l. Prove that B, B' are symmetric points with respect to line l.

5. $ABCD$ is trapezium in which $AB \parallel DC$. The line m is the perpendicular bisector of sides AB and CD. Prove that line m is an axis of symmetry of quadrilateral $ABCD$.

(Section—C)

6. If A, A' and B, B' are two pairs of points, each pair symmetric with respect to a line m, and AB is parallel to m, show that $A'B'$ is also parallel to line m.

□ □

19

AN INTRODUCTION TO THE COMPUTER

We are living in **'Computer Era'**. Computer is one of the most powerful machines ever designed by man to solve any type of problem. Computers are "smart" enough to play games, help teach students and solve difficult problems that would take a person hours, or even years, to solve. But inside, computers are really quite simple minded. When you come right down to it, they can understand only two things : On and off. A computer makes up for its limited vocabulary by being very, very fast. For example the time it takes you to read this paragraph, a typical micro or personal computer can check the spellings of every word in this chapter or even more. It won't learn anything, but if you give it the right instructions or input, it will give you the correct results or output.

So, in reality a computer is an electronic device which can perform computations, including arithmetic and logic operations, without intervention by a human being. It can also perform Input/Output functions and store information. But, one point should be clear that computer cannot do anything by itself as it does not have the 'brain' or thinking power. However, we can make computer do any type of processing by giving appropriate instructions.

MATHEMATICS AND COMPUTERS

One may wonder as to what computers have to do with mathematics. We note that three great names associated with the development of computers are those of mathematicians. These mathematicians are Charles Babbage (1791-1871), Alan Turing (1912-1954) and John Von Neumann (1903-1957).

Babbage gave the basic idea of a Computer. All the ideas, that Babbage proposed to be incorporated in his machine, are present in today's computers. Thus Babbage can be rightly considered as the father of computers. Turing was a logician-mathematician of Britain who did a lot of fundamental mathematical work in the area of computation. The architecture of the present day computers is the one designed by Von Neumann. Today's computers are based on the same architectural patterns and have only grown in their capacity to store more data and perform operations with greater speed.

If we go through the history of computing we will find the earliest processing that man required to do was arithmetical. The first instrument that was used for helping man in making calculations was the Abacus. Chinese have been using it for big calculations around 1200 A.D.

John Napier (1550-1617), a Scottish mathematician, invented logarithms. Napier's logs became the basis for a well-known computing machine, known as "Slide Rule". Napier's invention was significant contribution to the development of computing devices.

Blaise Pascal, a Frenchman, invented a machine in 1642 made up of gears which was used for adding the numbers quickly. We still use adding machines based on the same principle as Pascal's. Later on in 1694, Gottfried Leibnitz, a famous German mathematician improved on Pascal's machine. He developed a calculator which was able to perform all the four basic mathematical operations. These machines can be considered as ancestors of later day calculators and the present day computers.

We can now appreciate that a computer is basically a child of mathematics and mathematicians.

WHAT COMPUTERS CAN DO ?

Computer is an electronic machine, that stores and processes data, whether numeric or non-numeric.

Computers can perform only three basic functions : making arithmetic calculations (addition, subtraction, multiplication and division) ; comparing or testing two values ; and storing and retrieving data. Computers can perform these tasks only when given an appropriate set of instructions to follow.

Using the above mentioned basic functions computers can process the payroll for a business, set the type for a publishing company, maintain traffic control, print bills mathematically, prepare result in a school, assist in the education process, assist in medical research and patient diagnosis, check income tax returns, make railway and air reservations, guide commercial aircrafts, play chess, assist in designing of automobiles, etc.

The wide variety of computer applications is indeed impressive but let us not be blinded into thinking that computers can do everything. Computers are, after all, nothing more than machines in need of human instructions.

WHAT COMPUTERS CAN'T DO ?

Although a computer appears to possess tremendous power, we must keep in mind that a computer can't do anything unless it receives a set of instructions. Quite simply a computer lacks the very important human ability to think and reason out problems. Man might one day make such 'thinking computer' but human thinking and creativity can never be developed in a machine. A computer accepts data from us, processes it, and then communicates the results back to us. But only the human users of the machine are capable of understanding these results. Computers do not possess any independent intelligence.

Another factor which needs consideration is that a computer does not always function correctly. Basically, this is due to three kinds of problems : an error in the computer program, incorrect input or a machine malfunction.

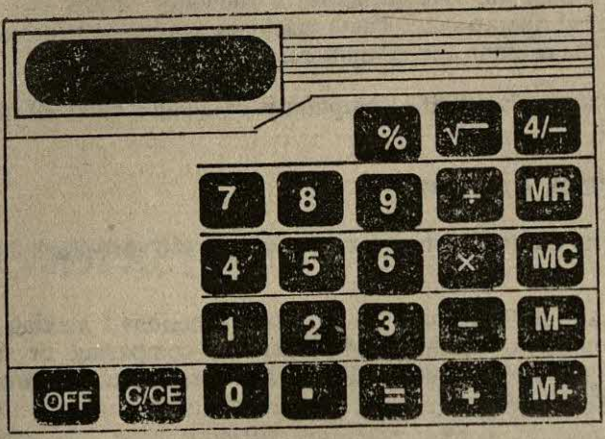
Since a computer is controlled by its program, it will do what the program instructs it to do. If wrong instructions are given, computer will follow wrong instructions faithfully giving incorrect results.

Errors can also result from incorrect input. This concept is referred to as 'Garbage in—Garbage out'.

Even when the program is error free, and the data entered is correct, the computer can still output faulty results. Since computer is an electronic device, it can easily be affected by temperature, humidity or any type of electrical disturbance etc.

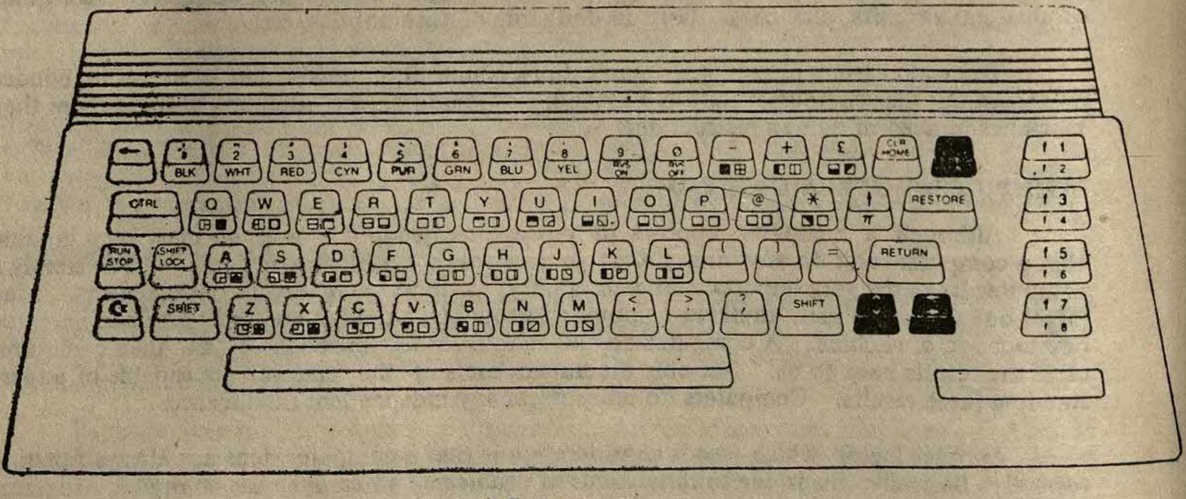
HOW IS A COMPUTER DIFFERENT FROM A CALCULATOR ?

You must have seen a calculator and even used it. It has a set of number keys and some function keys with a small display screen on the top. A typical calculator looks like this :



A Calculator

Do you know what a computer looks like ? It looks somewhat like this :



A Computer

In appearance it is just a bigger version of a calculator. It has a set of keys which resemble a typewriter and a screen which resembles a television screen. In a calculator the result of any problem comes on a small screen at the top. With a computer you have a much larger place to write on. You can write on the whole screen. But this is not the only difference. Let us know about some more important differences between a computer and a calculator.

A computer is an electronic device which is capable of accepting input, performing arithmetic and logic functions, storing information and giving output at a very fast speed and with great accuracy. Calculator is also electronic, works very fast, never tires and makes no mistake. But one very important difference between calculator and computer is that with a calculator you can only process or work with numbers, while with a computer you are able to work both with numbers and words.

A calculator cannot store instructions whereas a computer can. Once the instructions are stored in computer, it does what it has been instructed to do automatically, without any intervention by a human being. Whereas in a calculator all the instructions have to be given manually, one-by-one. In other words we can say that computers are programmable where as calculators are not.

Though, Japan has been able to make programmable calculators, but their scope is limited. Calculators still lack in their capability to store information permanently. With computers auxiliary memory devices or external memory devices can be used to store information, data programs etc., permanently. These devices cannot be used with a calculator. Thus backing storage devices such as magnetic tape, magnetic disks, floppy disks etc., provide computer with unlimited memory.

Thus we see that computers definitely have an edge over calculators or any other electronic device.

The spate of innovations and inventions in computer technology during the last decade has lead to the development of mini, micro and personal computers. They are so versatile that they have become indispensable to engineers, scientists, business executives, managers, accountants, teachers and students. They have strengthened man's power in numerical computations and information processing.

Modern computers possess certain characteristics and abilities that are peculiar to them. Now we shall discuss some of them.

CHARACTERISTICS OF COMPUTERS

1. SPEED

Computer is a very high speed information processing machine. The speed of the computer is due to the incredible speed of electrical pulses or movement of electrons. We do not talk of its speed in terms of seconds or even milli seconds but units of time used are microseconds (millionths), or nano seconds (thousand-millionths) and even the pico-seconds (million-millionths).

1 Second=1,000 milli-seconds

1 Second=1,000,000 micro-seconds

1 Second=1,000,000,000 nano-seconds

1 Second=1,000,000,000,000 pico-seconds

A powerful computer is capable of adding together two 18 digit numbers in 300 to 400 nano-seconds or we can say that 3 million calculations can be done in one second.

It is this high speed of the computer which has led to the completion of various scientific projects and achievements which otherwise were impossible *e.g.*, the control of moon landing would have not been feasible without the use of computers.

2. STORAGE

As a human acquires new knowledge the brain subconsciously selects what it feels is important and relegates unimportant details to the back of the mind. Though the memory of a computer is large enough to retain a great length of information, but this is also finite. Auxiliary memory devices further extend the scope of storing a wide range of information.

3. ACCURACY

Computer's accuracy is consistently high. Errors in the machinery can occur, but due to increased efficiency in error detecting techniques these seldom lead to false results. Although without exception, the errors in computations are due to humans rather than to the machine or its technological weakness. Just like an error in a typed letter is due to the fault of the typist and not the typewriter.

4. VERSATILITY

As computers are capable of performing almost every task, so many daily activities can be performed by computers. By using the craft of programming, a given problem can be solved using various operations, which make computers highly ingenious. Computers now-a-days are used in every field of life. It finds its application in reservation in Railways and Airlines, in Banks, in Examinations for checking papers etc, and in keeping accounts up-to-date in Business etc. This is the only machine which can do such a wide variety of jobs through software development.

5. AUTOMATION

A Computer is much more than an adding machine or a calculator which requires human operators to press necessary keys for operations to be performed. Once a program is in the computer's memory, the individual instructions are then transferred, one after the other, to the control unit for execution. No human assistance is needed after the program goes for execution.

6. DILIGENCE

Being a machine, a computer does not suffer from the human traits of tiredness and lack of concentration. If 3 million calculations have to be performed, it will perform all of them with exactly the same speed and accuracy.

PROBLEM SOLVING

Computers have provided man with the capability to handle many varied problems, which are tedious and routine in nature. Today there is hardly any field where computers are not used. They are popularly, though mistakenly, perceived to be electronic brains having the attributes of intelligent beings. They only help human beings in solving problems and decision-making. They do exactly what they are told ; they have no common sense. In every instance of problem solving, the computer cannot be used to solve the problem until a method of solution has been evolved and a detailed procedure has been prepared by the user.

Now the question arises, how to program a computer to solve a particular problem.

Every problem has some information given and some information is to be obtained. The question is how to process the given information so that the required results are obtained.

Problem solving on a computer is the task of expressing the solution to the problem in terms of simple concepts and operations and obtaining the results. The procedure to do this is as follows :

(*a*) Studying the problem in detail. Identifying output requirements, input data available and conditions and constraints to be used.

(*b*) Preparing list of procedure or steps necessary to arrive at the solution. This is called Algorithm.

(*c*) Preparing a flow chart. Flow chart is a diagrammatic representation of the procedure for solving the problem.

(*d*) Developing a program and running it on the computer.

Before we proceed any further, we should understand the term 'Algorithm' and its relation with mathematics.

In order to solve a problem we have to prepare a step-by-step procedure. A technical term used for step-by-step procedure is algorithm. So an algorithm specifies a sequence of steps which, when faithfully carried out will result in the job being completed.

In our day-to-day life we use many terms which are equivalently used for algorithms. For example in the subject of cookery, step-by-step procedure for preparing a certain dish is called by the term "recipe".

In many magazines you must have come across step-by-step procedures for preparing a pullover. These instructions put together are known as "**knitting pattern**". Similarly step-by-step procedure for preparing a dress is known by the term "**dress-pattern**". You must have played with '**Mechanno**'. You must have made a model crane or a cart using different building blocks. You have to follow certain instructions in a definite order to achieve your objectives. They are called "**assembly instructions**". We can summarize in a table, the terms used for the algorithm or step-by-step procedure to accomplish the task.

Task	*Common name*
Preparing a dish	recipe
Knitting a pullover	knitting pattern
Preparing a dress	dress pattern
Building a toy	assembly instructions

Table : Algorithms of some common tasks

Thus you can see, that preparing a step-by-step procedure is very important in order to solve any problem. Thinking process is involved in writing an algorithm. Thus learning how to develop an algorithm is learning how to think properly in a systematic fashion.

MATHEMATICS, ALGORITHM AND COMPUTERS

Algorithms are of great importance in solving any mathematical problem. Be it a problem of calculating average, calculation of LCM, calculation of HCF or calculation of profit, loss etc.

The basic idea behind writing an algorithm is that one should be able to communicate the method of solving the given problem to others effectively. It is not sufficient that a human being understands an algorithm himself but he should be able to communicate it to the computer such that the computer can solve the problem by following the given steps. A computer can handle only simple operations like addition, subtraction, multiplication, division and comparisons.

So the algorithm should be in terms of these simple operations only. The computer can solve only those problems for which algorithm can be written in terms of basic operations specified earlier. A computer cannot solve a problem for which algorithm cannot be written. But algorithm cannot be communicated directly to the computer. Once the algorithm is ready, it can be translated into any one of the languages that a computer can understand. Some of the examples of computer languages are BASIC, FORTRAN, COBOL, PASCAL etc. An algorithm written in any such language is called a **program**. So the algorithm is an aid in programming. Once the algorithm is ready, the sequence of steps is clear, it is very easy to code the algorithm in programming language. Thus, writing algorithm involves systematic thinking and helps in developing reasoning power and initiate systematic thinking process.

Now that you have understood the term algorithm, and its relation with mathematics, let us take the following examples for better understanding :

Example 1. Evaluate

$$10-8\times 5+(60/10)-(-10)+1$$

Solution. The problem is very simple and you may give the final answer quickly. But here, we are more interested in knowing the steps involved in solving the above problem and the order in which the steps are to be carried out. For evaluating the above expression we will have to keep in mind the rules and order of operation. Perhaps you would proceed step-by-step as follows :

Step 1 : While solving any expression negation is solved first. So the given expression becomes :

$$10-8\times 5+(60/10)+10+1.$$

Step 2 : Expression within the brackets, if any, are solved next in any expression. In this expression also brackets are solved for the values (60/10) and the two-factor term is reduced to one value. The given expression becomes :

$$10-8\times 5+6+10+1.$$

Step 3 : After all the brackets are solved and exponent values have been calculated, multiplication and division are performed (whichever comes first is solved first).

Now, in this expression multiplication is performed and the expression becomes :

$$10-40+6+10+1.$$

Step 4 : After all the brackets have been removed, multiplication and division have been performed, addition and subtraction are performed from left to right.

In this expression minus sign is to be performed first.

Note that after the operation, the expression will become :

$$-30+6+10+1.$$

Step 5 : Similarly the expression is solved further. Thus giving the expression

$$-24+10+1.$$

Step 6 : Expression is reduced to

$$-14+1.$$

Step 7 : The final value that we get is

$$-13.$$

Notice that the answer is negative.

Let us now consider a some what general situation.

Example 2. Evaluate

$$a\times b-(c)+d/f+(-h).$$

Where a, b, c, d, e, f and h are numbers and $f<>0$.

Solution. The method of evaluating this expression will be similar. The only difference will be that here we are using symbols a, b, c, etc., instead of actual specific numbers. Try to write down the step-by-step procedure yourself using numbers for various symbols.

Example 3. Obtain total marks if marks in two subjects are given.

In this, information is given about marks in two subjects and the processing that is to be done with those marks, *i.e.*, to add them.

Thus we see that, to solve any problem a procedure is needed, but if that problem is to be solved by a computer that procedure should be a step-by-step procedure. Everything in the procedure should be clearly and properly defined such that if that procedure is followed by a person blindly, he should be able to solve the problem.

The procedure written above should be more precise and clear. Algorithm can be written for the above problem as follows :

Say A and B are marks in two subjects. Then total is given by the formula :

$$T=A+B.$$

Algorithm

```
(1) Read marks in two subjects and call them $A$ and $B$ respectively.
(2) Add $A$ to $B$ and call it TOTAL.
(3) Write the value of TOTAL.
(4) Stop.
```

Now let us take one problem from mathematics and write the algorithm for it.

Example 4. Calculate the area of a triangle when its three sides are given.

Say a, b, c are the sides of the triangle

$$\text{LET } s=(a+b+c)/2. \quad \text{...(1)}$$

Then area (X) is given by

$$X=\sqrt{s(s-a)(s-b)(s-c)} \quad \text{...(2)}$$

Algorithm

```
1.  Read values of sides $A$, $B$ and $C$.
2.  Add $A, B, C$ and call their sum $D$.
3.  Divide $D$ by 2, and call it $S$.
4.  Subtract $A$ from $S$ and call it TEM1.
5.  Subtract $B$ from $S$ and call it TEM2.
6.  Subtract $C$ from $S$ and call it TEM3.
7.  Multiply $S$, TEM1, TEM2, TEM3 and call it $Y$.
8.  Find square root of $Y$ and call it $X$.
9.  Write $X$.
10. Stop.
```

Note : There is a SQR function available with the computer to calculate the square root of the given value so utilize that function to calculate square root.

The above procedure is useful and practically available for solving complete problem and developing large program. One thing should be noted that algorithm can not be entered in the computer as it is not a program. Rather algorithm is a tool in programming, and it is a step towards the program.

Some simple problems from our day-to-day life are given and step-by-step procedures are also given to solve them.

Example : Let us write a step-by-step procedure for crossing the road.

Step 1 : Go to the edge of the road.

Step 2 : Look at your right hand side.

Step 3 : If there is any vehicle on the road then go to step 2 otherwise go to step 4.

Step 4 : Cross half of the road and stop in middle.

Step 5 : Look at your left hand side.

Step 6 : If there is any vehicle on the road then go to step 5 otherwise go to step 7.

Step 7 : Cross the road.

Step 8 : Stop.

In fact this is how you teach a child to cross the road. This step-by-step procedure works well with many people in many situations. However, this procedure does not take into consideration many more possibilities that may arise. For example, if some vehicle is parked on the road side, then the person who wants to cross the road, will keep on looking at the right-hand-side because step 3 does not specify whether the vehicle is moving or stationary. So, this procedure has to be rewritten to take into consideration the situation when a vehicle is just parked on the road.

Step 1 : Go to the edge of the road.

Step 2 : Look at your right-hand-side.

Step 3 : If there is no vehicle on the road go to step 6 otherwise go to step 4.

Step 4 : If vehicle is parked by road side then go to step 6 otherwise go to step 5.

Step 5 : If moving vehicle is atleast 50 metres away from you then go to step 6 otherwise go to step 2.

Step 6 : Cross half the road.

Step 7 : Look at your left hand side.

Step 8 : If there is no vehicle on the road go to step 11 otherwise go to step 9.

Step 9 : If that vehicle is parked by the road side then go to step 11 otherwise go to step 10.

Step 10 : If moving vehicle is about 50 metres away from you then go to step 11 otherwise goto step 7.

Step 11 : Cross the road.

Step 12 : Stop.

Even this step-by-step procedure doesn't take into account many more possibilities, such as a zebra crossing nearby or a flyover for crossing the road or a one-way street. So this procedure can be changed further to take into consideration the various other possibilities.

Example : You want to telephone your friend to give him/her some information. Write step-by-step procedure such that while applying that procedure, every one, who wants to telephone, is able to do so.

Step 1 : Take the receiver in your hand and dial the telephone number of the person whom you want to phone.

Step 2 : If the ring goes, go to step 3 otherwise go to step 7.

Step 3 : If some one speaks from the other side, verify the number you have dialled. If no one speaks then go to step 8.

Step 4 : In case the number is correct, talk to your friend or get the information about him. In case the number is wrong go to step 9.

Step 5 : Keep the receiver on telephone.

Step 6 : Stop.

Step 7 : The telephone on the other side seems to be engaged. Keep the receiver on the telephone for a second and go to step 1.

Step 8 : No one seems to be at home. Go to step 5.

Step 9 : It seems that you have got wrong telephone number. Say "SORRY". Keep the receiver on telephone. Go to step 1.

This step-by-step procedure for telephoning any one will work in most of the cases but if the person who wants to telephone does not know the telephone number a query takes place in step 1 *i e.*, step 1 cannot be performed as in step 1 dialling of the telephone number is needed and that number is not known. So there is a need to revise this procedure to handle these type of situations. This is how the procedure is improved to handle more general problems. The main aim is that the procedure should be such that no thinking is required while performing the steps of the procedure as these procedures are to be performed by the computer (electric machine) which cannot think but can faithfully follow the instructions.

The above procedure is now written again to include those situations when one wants to telephone but does not know the telephone number.

Step 1 : Do you know the telephone number of your friend ? If yes go to step 2 otherwise go to step 12.

Step 2 : Take the receiver in hand and dial the telephone number.

Step 3 : If the ring goes, go to step 4 otherwise go to step 8.

Step 4 : If some one speaks from the other side, verify the number you have dialled. If no one speaks go to step 10.

Step 5 : In case the number is correct, talk to your friend or get the information about him. In case the number is wrong go to step 11.

Step 6 : Keep the receiver on telephone.

Step 7 : Stop.

Step 8 : The telephone seems to be engaged.

Step 9 : Keep the receiver on the telephone for a second and go to step 2.

Step 10 : No one seems to be at home, go to step 6.

Step 11 : It seems that you have got wrong telephone number. Say "SORRY". Go to step 9.

Step 12 : Do you have the telephone directory ? If not go to step 14.

Step 13 : Look for the name in the Telephone directory. The names in the telephone directory are arranged alphabetically. If you have got the name, note the telephone number ; go to step 2 otherwise go to step 14.

Step 14 : Ring telephone enquiry and ask the telephone number giving the name. If you get the telephone number, note that number in your diary for future reference and go to step 2. In case you do not get the telephone number go to next step.

Step 15 : It appears that the telephone number is not in the name of that person. Dial again the telephone enquiry number and ask for the telephone number giving the address of that person whom you want to phone. In case you do not know the address then go to step 6, otherwise proceed to next step.

Step 16 : In case you get the telephone number against address ; go to step 2 ; otherwise go to step 6 as there is no telephone in the residence of that person whom you want to phone.

Thus we see the second step-by-step procedure is more general than the first one. This algorithm procedure is performed step-by-step *i.e.*, first step 1 is performed, then step 2. and so on. In case we do not want next step to be processed then statement like go to is used. Then, immediately from that step one goes to the step which is mentioned in go to statement ignoring the steps in between. For example in the above second procedure, if the telephone number is not known, it asks us to proceed to step 12 ignoring steps 2 to 11. After going to step 12 the steps will be again processed in the sequential order unless the procedure is again disturbed by a go to statement.

We note that the above procedure can be performed by a child who simply understands English.

STEPWISE REFINEMENT OF ALGORITHMS

Computer is a very obedient machine. It carries out the instructions most honestly and it performs operations exactly as per the instructions given to it. Therefore every step given should be very precise, correct and non-confusing. For that purpose, our algorithms require successive refinements until each of its step is sufficiently detailed, exact and precise. This approach is known as step-wise refinement. Let us take one example to explain this approach.

Example : Design an algorithm to describe how to make rice for meals for a person.

Solution :

The required logarithm will consist of following steps :

Step 1 : Pick the rice.

Step 2 : Wash the rice with clean water 2 or 3 times.

Step 3 : Put the rice in boiling water.

Step 4 : Add salt to it.

Step 5 : Cook till done.

Step 6 : Serve rice in plate.

The steps given above are not sufficiently detailed to be carried out successfully by a computer. Each step requires further breaking down into simple sub-steps, with each sub-step specifying more details than the original step.

Step (1) Pick the rice can be refined into :

1.1 Open the rice tin
1.2 Measure 1/2 glass of rice
1.3 Put rice in a plate
1.4 Remove stones, etc., from the rice.

Step (2) Wash the rice can be refined as :

2.1 Put rice in a bowl
2.2 Take 1 glass water from the tap
2.3 Pour water in the bowl
2.4 Wash the rice to remove dirt etc.
2.5 Drain dirty water
2.6 Repeat process 2.3 through 2.5 until all dirt gets removed.

Step (3) Put the rice in boiling water can be refined as :

3.1 Take one and a half glass of water
3.2 Pour the water in an empty vessel
3.3 Light the gas
3.4 Keep the vessel on the stove
3.5 Wait until water starts boiling
3.6 Put rice in boiling water.

Step (4) Put salt in rice can be refined as :

4.1 Open masala box
4.2 Take half spoonful of salt

4.3 Put the salt in boiling water
4.4 Close the masala box.

Step (5) Cook till done can be refined into :

5.1 Take a lid
5.2 Put the lid on the vessel
5.3 Slow down the flame
5.4 Cook for 7 minutes
5.5 Switch off the gas
5.6 Keep the vessel covered for another two minutes.

Step (6) Serve rice in the plate. This step can be refined as :

6.1 Take a plate
6.2 Take a rice spoon
6.3 Put rice spoon in plate
6.4 Take a serving spoon
6.5 Uncover the vessel
6.6 Put two serving spoons full of rice in the plate
6.7 Serve it.

Thus refinement of a step will involve :

(*a*) breaking down the present step of action into small and simpler steps of action.
(*b*) re-expressing the present step of action with more precision or
(*c*) adding more details by way of sub-steps.

One thing should be clear that refinement of steps need not increase the number of sub-steps if the problem is very simple and within the grasp of an individual. One can come up with the entire algorithm in one stroke without the requirement of further refinement.

Now you can answer the question, which we asked earlier : How does a human being attempt to tackle a problem ? You must have understood something about it by now from the examples given. Whenever anybody is confronted with a problem, his understanding of the problem makes him think the broad steps that will lead to the solution. The person may not be able to think the minute details of the steps involved but eventually when the person takes steps one-by-one he may add the details by breaking the given step into a number of sub-steps. This approach to problem solving is known as **step-wise refinement.**

EXERCISE 19.1

1. Evaluate the following expressions using step-by-step approach :
   (*a*) $(-12)/(-2)+8-(6\times 10)-(-25)$
   (*b*) $-(-20)-(5\times 4/2)+5\times 7$
   (*c*) $23-123+6\times 49-12/4-(-10)$
2. Develop an algorithm to prepare a cup of tea.
3. Design an algorithm to read a book.
4. Workout the detailed steps for searching the phone number from a directory.
5. We want to devise an algorithm to borrow a book from a library. Following is the initial algorithm with broad steps :
   1. Find the number assigned to the book from the catalogue.
   2. Ascertain whether the book is available on the shelf of the library.

3. If available, get the book issued.

Refine these steps further. Incorporate the following while refining it.

(*a*) If the book is not held by the library, recommend it for purchase.

(*b*) If the book is held by the library but currently not available put it on demand slip.

6. Develop an algorithm to find smaller of two numbers.
7. Develop an algorithm to find smallest of 3 numbers.
8. Design an algorithm to calculate the area of a circle for given radius '*r*'.

FLOW CHARTS

We have seen that to use computers for solving problems, it is necessary to develop a step-by-step procedure. These steps have to be serial *i.e.*, one after the other, because the computer operates serially. These procedures can be represented in another way through figures called **flow charts**.

A flow chart is a pictorial representation of the steps involved in the procedure. It also shows the logical sequence, in which the steps are performed. A flow chart consists of boxes, called symbols and arrows, called flowlines. The box tells us what to do and the flowline shows us the next step to be done.

In flow charts each step is put into a box to emphasize that it is an individual unit and lines indicate the flow of sequence. For example you want to calculate the total marks for a student. Marks in three subjects are given and you call them by names *A*, *B*, *C*. So for solving this problem, following step-by-step procedure can be used.

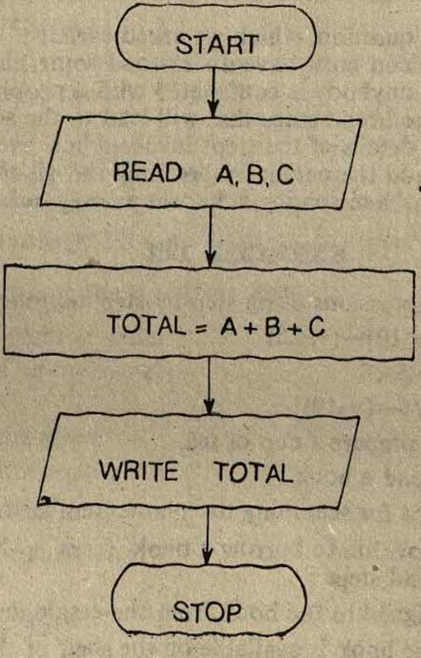
Step 1 : Read marks in three subjects. Call them *A*, *B*, *C* respectively.

Step 2 : Add *A*, *B*, *C* and call it TOTAL.

Step 3 : Write TOTAL.

Step 4 : Stop.

Once the problem has been broken into steps and precise sequence of steps is decided, the programmer can illustrate each individual step through the use of a flow chart.

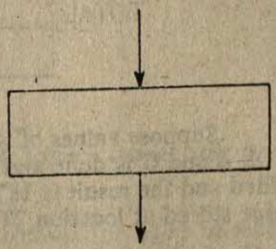


(Here *A*, *B*, *C* represent memory cells which can hold any value.)

The above algorithm has the following steps ; obtaining the input data, performing calculations and producing the output information. Each type of step is shown using a different shape of box. In above flow chart you can notice following types of boxes :

Processing Box

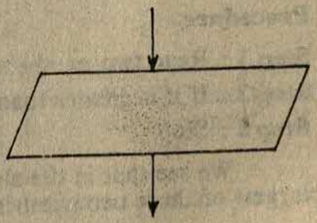
The shape of the processing box is a rectangle. This symbol represents one or more instructions which perform a processing function of the program. Examples of processing functions are addition, subtraction, multiplication, division, moving data in storage or initializing the value of some variable as in case of assigning the value zero to sum *i.e.*, SUM=0 at the start or formula like TOTAL=$A+B+C$.



Only one flow line can move out of the processing box and any number of lines can enter it.

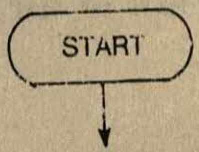
Input/Output Box

The shape of input/output box is a parallelogram. This symbol indicates any function of an input/output device such as keyboard or printer. It makes data available for processing or displaying the result of processing on screen. Examples of Input/Output operations are keying in data through key-board or displaying some information on the screen or printing it on printer.



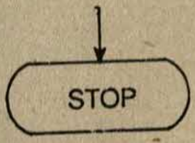
Again only one flow line can leave this box and any number of flow-lines can enter this box. In the above example Read *A*, *B*, *C* and Write TOTAL are input/output box respectively.

Terminal Box

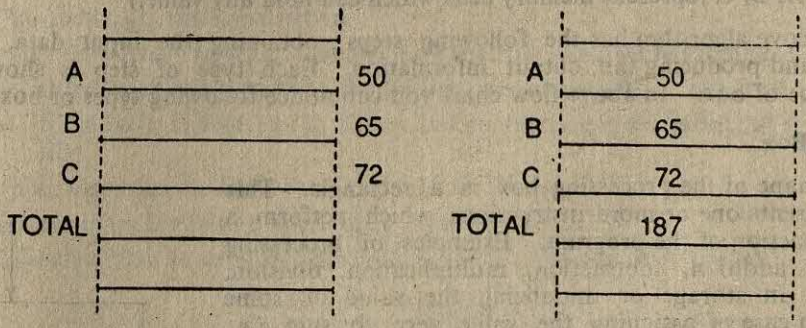


This symbol is used to indicate the beginning or end of a flowchart. When this symbol is used for START, no flow line can enter it. Only one flow-line can leave from START box. When it is used for STOP, no flow line can leave this box. Any number of flow-lines can enter STOP box.

There is only one START symbol in each flowchart, however there can be a number of STOP symbols in a given flowchart.



Let's analyze what happens in the computer's memory. Computer memory can be thought to be made up of small cells, where different values can be stored. READ *A*, *B*, *C* in box No. 2 reserves memory space and stores the values in boxes named *A*, *B* and *C* which can have any numeric values.



Suppose values of *A*, *B* and *C* are 50, 65 and 72 respectively. In box No. 3, addition of *A*, *B* and *C* is done and is stored at memory cell named TOTAL *i.e.*, 50, 65 and 72 are added and the result is 187 which is stored at memory location TOTAL. In box No. 4 the value stored at location TOTAL is printed and box No. 5 indicates end of procedure.

Example

Write a step-by-step procedure for finding the largest of 2 numbers.

Procedure

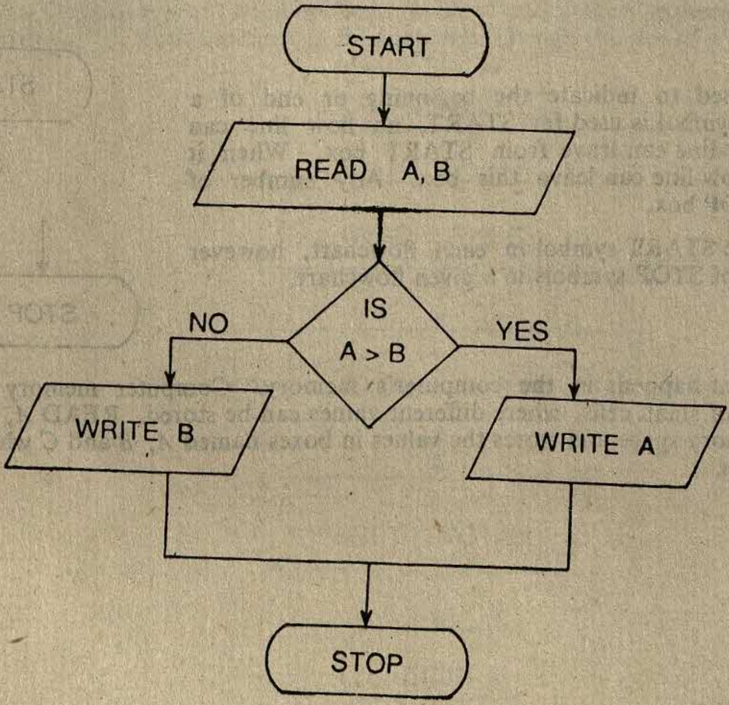
Step 1 : Read two numbers and call them *A* and *B* respectively.

Step 2 : If *A* is greater than *B* then write *A* otherwise write *B*.

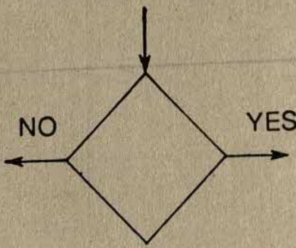
Step 3 : Stop.

We see that in the above procedure you can read two numbers and it will print the largest of those two numbers. The result will be obtained in step 2.

The above procedure can be pictorially represented as :



Here we find the use of another symbol *i.e.*, **Decision Box**



It resembles a diamond in shape. This box is used when some comparison is to be performed between two quantities.

So a decision symbol shows in the procedure where a comparison is to be made and depending on the value of condition, one of the two different steps is to be taken. Decision box gives rise to two alternative answers to the condition *i e.*, either the condition is true or it is false. In a decision box following conditions can be checked :

= < <= > >= < >

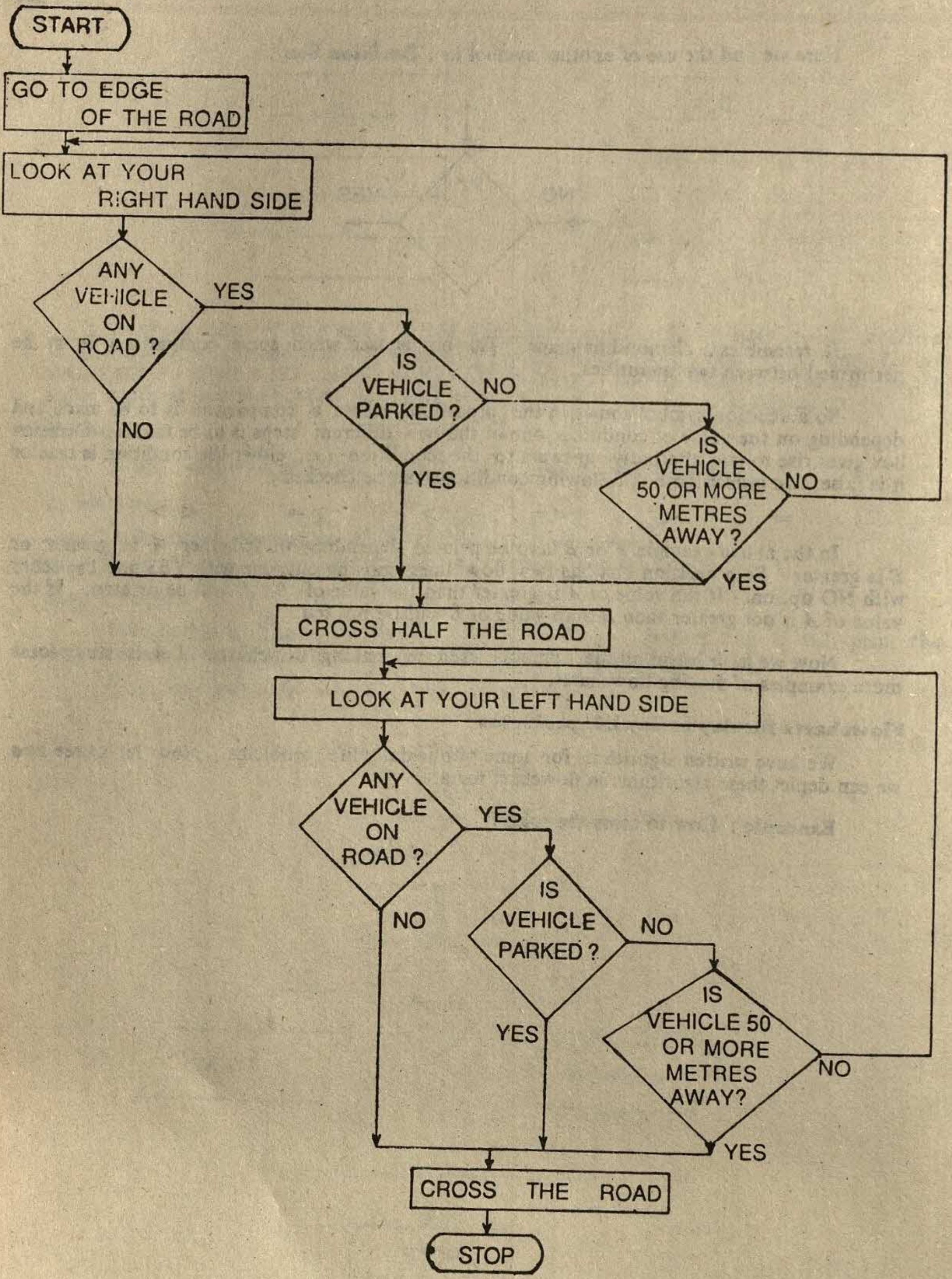
In the above example *A* or *B* is to be printed depending on whether *A* is greater or *B* is greater. So a decision box has two flow lines moving out, one with YES and the other with NO option. If the value of *A* is greater than the value of *B*, *A* will be printed. If the value of *A* is not greater than *B* than value of *B* will be printed.

Now we have learnt all the symbols used in making flowcharts. Let us study some more examples of making flowcharts.

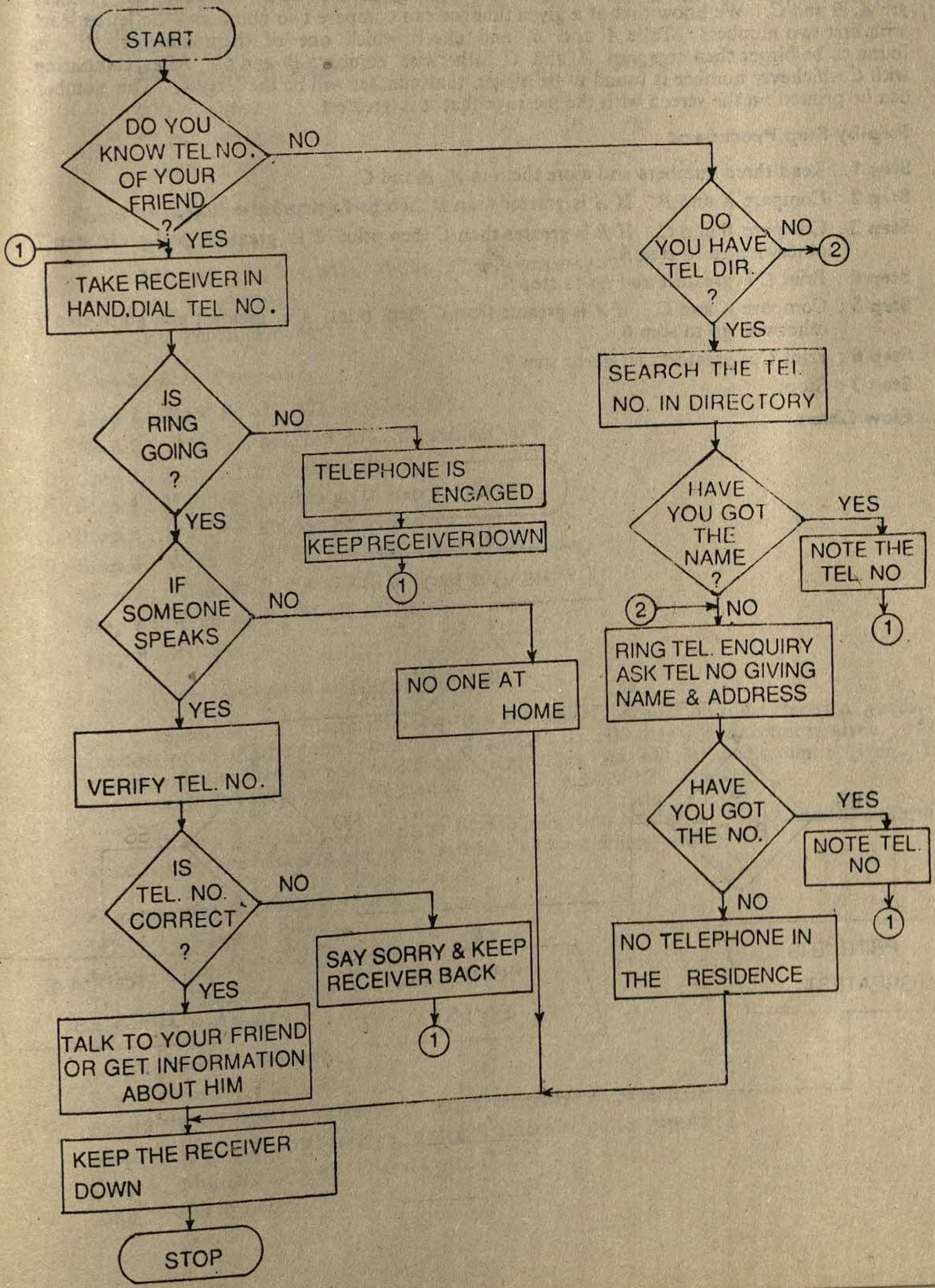
Flowcharts for day-to-day life problems

We have written algorithms for some of our daily life problems. Now let us see how we can depict these algorithms in flowchart form.

Example : How to cross the road ?


START
GO TO EDGE OF THE ROAD
LOOK AT YOUR RIGHT HAND SIDE
ANY VEHICLE ON ROAD ?
YES
NO
IS VEHICLE PARKED ?
NO
YES
IS VEHICLE 50 OR MORE METRES AWAY ?
NO
YES
CROSS HALF THE ROAD
LOOK AT YOUR LEFT HAND SIDE
ANY VEHICLE ON ROAD ?
YES
NO
IS VEHICLE PARKED ?
NO
YES
IS VEHICLE 50 OR MORE METRES AWAY?
NO
YES
CROSS THE ROAD
STOP

Example : Making a telephone call to your friend.



Example : To find out the largest of three numbers given to you.

Here we want to find out the largest of three given numbers. Suppose the numbers are *A*, *B* and *C*. We know that at a given time we can compare two numbers. So let us start with first two numbers. Take *A* and *B*, and check which one of them is bigger. If *A* is found to be bigger then compare *A* and *C*, otherwise compare *B* and *C*. After comparing with *C* whichever number is found to be bigger, that number will be the largest. The number can be printed on the screen with the message that it is greatest.

Step-by-Step Procedure

Step 1 : Read three numbers and store them in *A*, *B* and *C*.

Step 2 : Compare *A* and *B*. If *A* is greater than *B* then go to step 5 else step 3.

Step 3 : Compare *B* and *C*. If *B* is greater than *C* then print *B* is greatest and go to step 7 otherwise go to step 4.

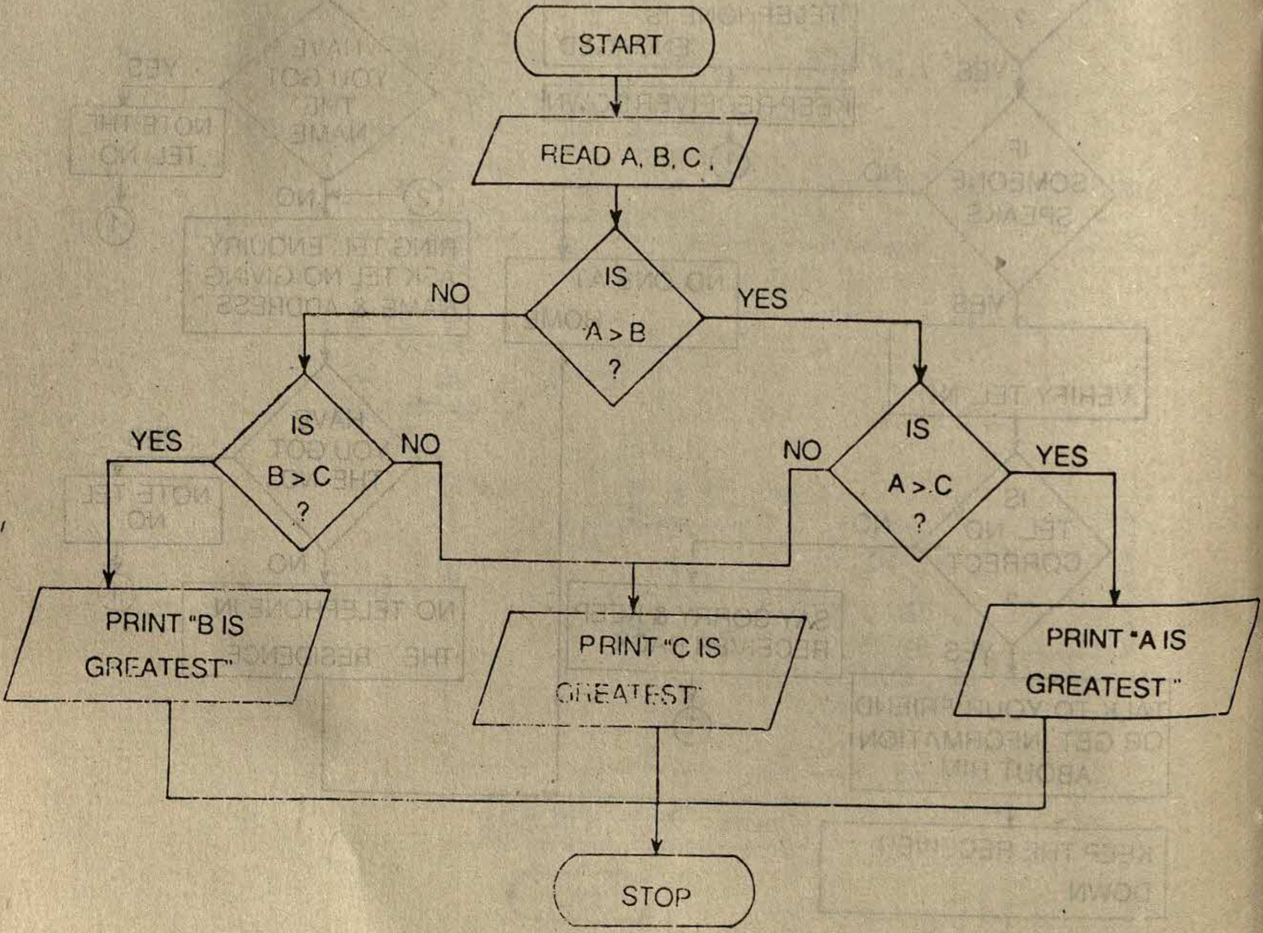
Step 4 : Print *C* is greatest and go to step 7.

Step 5 : Compare *A* and *C*. If *A* is greater than *C* then print *A* is greatest and go to step 7 otherwise go to step 6.

Step 6 : Print *C* is greatest and go to step 7.

Step 7 : Stop.

Flow Chart



Example : Draw a flowchart to compute and print the grades for an examination. Input is Roll No. and Marks in five subjects out of 100. Grades are allocated on the following basis :

Percentage of Marks	**Grade**
90 and above	'*A*'
80 and less than 90	'*B*'
Less than 80	'*C*'

Short names for memory cells

R : Roll No.
M1, M2, M3, M4, M5 : Marks in 5 subjects respectively.
T : Total marks.
P : Percentage marks.

Step-by-Step Procedure

Step 1 : Read R, M1, M2, M3, M4, M5.
Step 2 : Add M1, M2, M3, M4, M5 and call it T.
Step 3 : Divide T by 5 and call it P.
Step 4 : If P >= 90 then go to step 7.
Step 5 : If P >= 80 then go to step 8.
Step 6 : Write R, P, "C GRADE" and go to step 9.
Step 7 : Write R, P, "A GRADE" and go to step 9.
Step 8 : Write R, P, "B GRADE".
Step 9 : STOP.

(Flow chart given on page 338).

Example : A departmental store has announced discount on some electrical goods. If the cost price of the good is less than or equal to Rs. 300 then 7% discount is given. If the cost price is less than or equal to 500 but greater than Rs. 300 10% discount is given. For appliances with cost price greater than Rs. 500, discount of 12% is given.

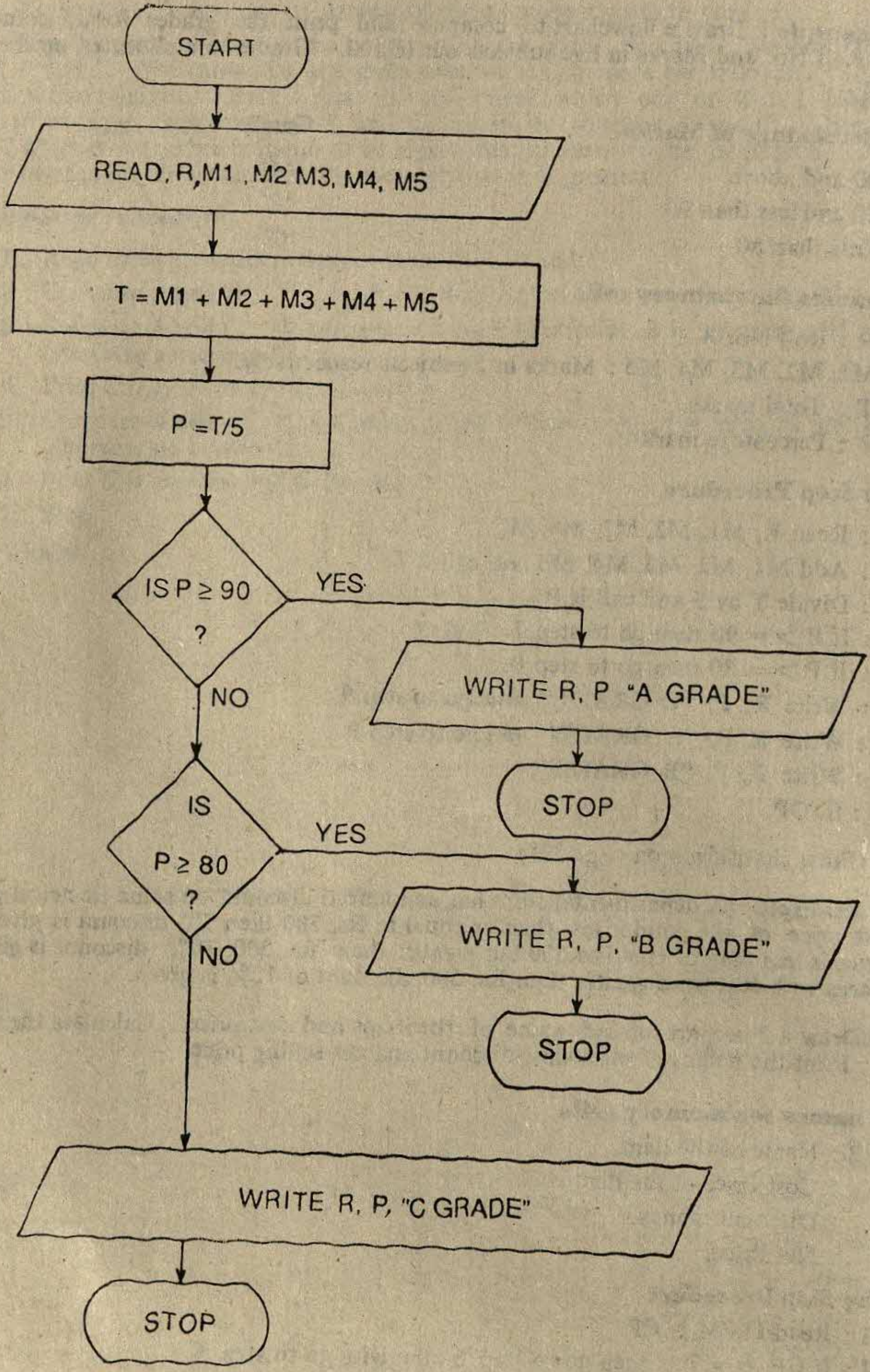
Draw a flowchart to read name of the items and cost price. Calculate the net selling price. Print the names of the items, discount and net selling price.

Short names for memory cells

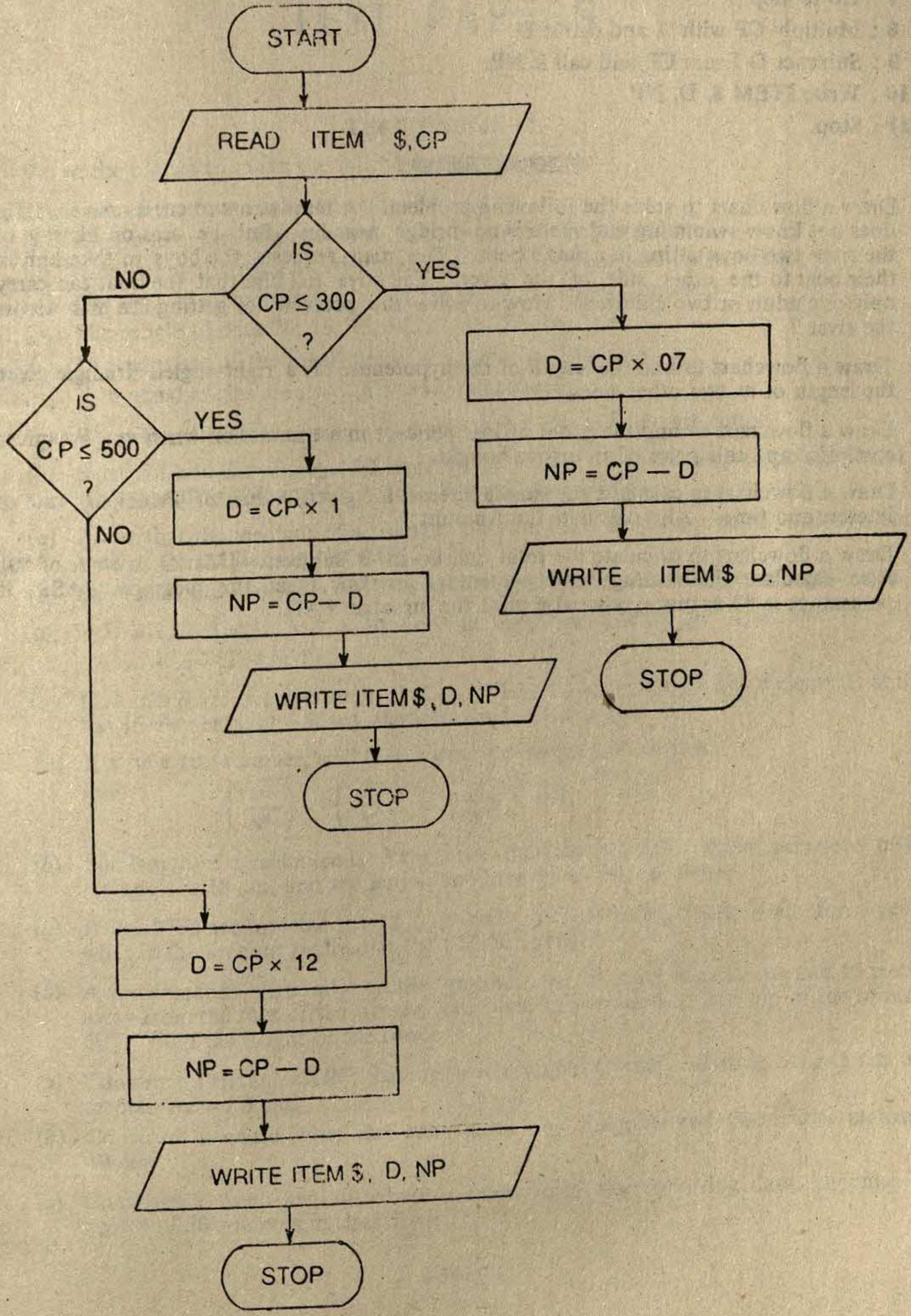
ITEM $: Name of the item.
CP : Cost Price of the item.
D : Discount money.
NP : Net Price.

Step-by-Step Procedure

Step 1 : Read ITEM $, CP.
Step 2 : If CP <= 300 then go to step 6 otherwise go to step 3.
Step 3 : If CP <= 500 then go to step 8 otherwise go to step 4.
Step 4 : Multiply CP with ·12 and call it D.
Step 5 : Go to step 9.


START
READ, R,M1 ,M2 M3, M4, M5
T = M1 + M2 + M3 + M4 + M5
P =T/5
IS P ≥ 90 ?
YES
NO
WRITE R, P "A GRADE"
STOP
IS P ≥ 80 ?
YES
NO
WRITE R, P, "B GRADE"
STOP
WRITE R, P, "C GRADE"
STOP

Flow Chart


START
READ ITEM $, CP
IS CP ≤ 300 ?
YES
NO
D = CP × .07
NP = CP — D
WRITE ITEM $ D, NP
STOP
IS C P ≤ 500 ?
YES
NO
D = CP × 1
NP = CP— D
WRITE ITEM $, D, NP
STOP
D = CP × 12
NP = CP — D
WRITE ITEM $, D, NP
STOP

```
Step  6 : Multiply CP with ·07 and call it D.
Step  7 : Go to step 9.
Step  8 : Multiply CP with ·1 and call it D.
Step  9 : Subtract D From CP and call it NP.
Step 10 : Write ITEM $, D, NP.
Step 11 : Stop.
```

EXERCISE 19.2

1. Draw a flow chart to solve the following problem. A man wants to cross a river. He does not know swimming and there is no bridge near by. But he sees on his side of the river two boys sitting in a small boat. The man requests the boys to take him in their boat to the other side of the river. The boys tell him that the boat can carry only one adult or two children. How to solve the problem of getting the man across the river ?

2. Draw a flowchart to find the length of the hypotenuse of a right-angled triangle given the length of its two other sides.

3. Draw a flowchart to find the profit or loss per cent in a transaction when you are given cost price and sale price of an article bought.

4. Draw a flowchart to calculate the simple interest for given value of Principal, rate of interest and time. Also calculate the Amount.

5. Draw a flowchart to calculate the total marks in 3 subjects. [Marks are out of 50]. Also calculate the Average and percentage marks. Print the message 'PASS' if percentage is 40 or more, otherwise print the message 'FAIL'.

TEST PAPERS

TEST PAPER—I

1. Fill in the blanks to make the following statements true :

   (*a*) Sum of the four angles of a quadrilateral is ....

   (*b*) A parallelogram is a rhombus if and only if its diagonals are ....

   (*c*) A polynomial having all coefficients as ....is called zero polynomial.

   (*d*) The set I of integers is a.........of the set *Q* of rational numbers.

   (*e*) The rangle of data 20, 15, 15, 14, 23, 18, 27, 16, 21 is..........

   (*f*) The mantissa of the logarithm of a number is always..........

   (*g*) Two equal angles are.........always symmetrical.

2. (*a*) Prove that the sides opposite to equal angles of a triangle are equal.

   (*b*) Sides *BA* and *CA* of a $\triangle ABC$ have been produced respectively to points *D* and *E* such that $BA=AD$ and $CA=AE$. Prove that $DE \parallel BC$.

3. (*a*) Prove that the medians of a triangle pass through the same point which divides each of the medians in the ratio 2 : 1.

   (*b*) Show that the diagonals divide a parallelogram into four triangles of equal area.

4. (*a*) If $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, find the complement of the set *A*, when
   $A=\{2, 4, 6, 8\}$

   (*b*) If *A* and *B* are two sets such that $A\cup B$ has 18 elements, *A* has 8 elements and *B* has 15 elements, how many elements does $A\cap B$ have ?

5. (*a*) If *x* be a real number, and *l*, *m*, *n* positive integers, show that

$$\left(\frac{x^m}{x^n}\right)^l \cdot \left(\frac{x^n}{x^l}\right)^m \cdot \left(\frac{x^l}{x^m}\right)^n = 1.$$

   (*b*) The length of a rectangle is 3 cm more than its breadth. If the perimeter of the rectangle is 18 cm, find the length and breadth of the rectangle.

6. (*a*) If Rs. 2730 are invested at 12% interest, compounded every year, for 5 years, what is the amount realised at the end of 5 years ?

   (*b*) A tower stands vertically on the ground. At a point on the ground 15 metres away from the foot of the tower, the angle of elevation of the top of the tower is 60°. Find the height of the tower.

7. (*a*) Construct a triangle *ABC*, having given angle $C=60°$, altitude $AL=5$ cm and median $BE=7{\cdot}5$ cm.

   (*b*) Construct a square when the sum of its one diagonal and two sides measures 10 cm

8. (*a*) Construct a frequency distribution table from the following data, showing the scores of 30 students in class test :

Marks	Number of students
Below 10	5
Below 20	12
Below 30	20
Below 40	26
Below 50	30

(*b*) In a study to test a new variety of wheat, an experiment was performed on 50 similar plots (under similar conditions) and the following results were obtained :

Yield per hectare (in quintals)	Number of fields.
10—20	2
20—30	7
30—40	12
40—50	15
50—60	8
60—70	6

(*i*) Construct a cumulative frequency table for the data.

(*ii*) Represent the above data by a histogram.

TEST PAPER—II

1. Fill in the blanks in the following so as to make each statement true.

(*a*) Triangles on the same base and between the same parallels are.........in area.

(*b*) Diagonals of a parallelogram.........each other.

(*c*) Every set is a .... of itself.

(*d*) If the class marks in a frequency distribution are 19·5, 26·5, 33·5, 40·5, 47·5, 54·5, 61·5, then class size is........ .

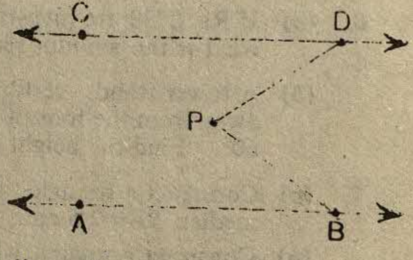
(*e*) 2 is a.........of the polynomial $4-x^2$.

(*f*) $3x^8+7x+5$ is a polynomial of degree.........in x.

2. (*a*) Prove that the sum of the three angles of a triangle is 180°.

(*b*) In the figure, lines *AB* and *CD* are parallel and *P* is any point as shown.

Prove that $\angle ABP+\angle CDP=\angle DPB$



3. (*a*) Prove that if a transversal intersects two parallel lines, then the interior angles on the same side of the transversal are supplementary.

(*b*) *ABCD* is a trapezium in which sides *AB* and *DC* are parallel, and *E* is the mid-point of *AD*. If *G* is a point on the side *BC* such that segment *EG* is parallel to side *DC*, prove that $EG=\frac{1}{2}(AB+DC)$.

4. (*a*) Find the remainder when the polynomial

$$p(x)=x^4-3x^2+2x+1 \text{ is divided by } x-1,$$

(*b*) In a class of 30 students, 20 students like to play cricket, and 15 like to play football. Also each student likes to play at least one game. How many students like to play both cricket and football ?

5. (*a*) Express $\frac{4\sqrt{3}+5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$ with a rational denominator.

(*b*) Factorise : $5x^2-32x+12$.

6. (*a*) Evaluate $2\sin^2 30^\circ \tan 60^\circ - 3\cos^2 60^\circ \sec^2 30^\circ$.

(*b*) Two men are on the opposite sides of a tower. They measure the angles of elevation of the top of the tower as 45° and 30° respectively. If the height of the tower is 40 m, find the distance between the men.

7. (*a*) Simplify : $\sqrt[bc]{\frac{x^b}{x^c}} \times \sqrt[ca]{\frac{x^c}{x^a}} \times \sqrt[ab]{\frac{x^a}{x^b}}$

(*b*) Divide the polynomial a^6-5a+4 by a^2-2a+1.

8. (*a*) Construct a cumulative frequency distribution table from the following frequency distribution table and find the number of students obtaining below 15 marks :

Marks	Frequency
0—5	4
5—10	12
10—15	10
15—20	8
20—25	3
25—30	2

(*b*) The distribution of monthly wages of 40 workers in a factory is given below. Draw a cumulative frequency curve for the data.

Monthly wages (In rupees)	Number of labourers
210—230	4
230—250	7
250—270	5
270—290	9
290—310	5
310—330	6
330—350	4

TEST PAPER—III

1. Fill in the blanks to make the following statements true :

(*a*) If two sides of a triangle are unequal, the longer side has the ....angle opposite to it.

(*b*) A quadrilateral is a parallelogram if and only if its diagonals ....each other.

(*c*) Every point of the number line corresponds to a........ number.

(*d*) The number of observations in a particular class interval is called the.. ......of the class interval.

(*e*) The sum of two quadratic polynomials is always a.........polynomial.

(*f*) A set and its complement set are always.........sets.

(*g*) The logarithm of 1 to any base is equal to.........

(*h*) The value of $\tan 45°+\cot 45°$ is.........

2. (*a*) Prove that parallelograms on the same base and between the same parallels are equal in area.

(*b*) ABC is a triangle in which $AC>AB$ and the bisectors of angles B and C intersect each other at O. Prove that $OC>OB$.

3. (*a*) Prove that in a triangle, the line segment joining the mid-points of two sides is parallel to the third side.

(*b*) Prove that the perpendicular bisectors of the sides of a triangle pass through the same point.

4. (*a*) If a real function f is defined such that its value at x is given by $f(x)=7-3x^2$, find the domain of f. Evaluate $f(1)$ and $f(-1)$.

(*b*) Factorise : $6x^2-x-2$.

(*c*) Show that 3 is a zero of polynomial $x^3-8x^2+8x+21$.

5. (*a*) If $\sin\theta=\frac{3}{5}$, find $\cos\theta$ and $\tan\theta$.

(*b*) As observed from the top of a 50 m tall light house, the angles of depression of two ships approaching it are 30° and 45°. If one ship is directly behind the other, find the distance between the two ships.

6. (*a*) Form a frequency distribution table for the following data :

Marks	Number of students
Above 0	30
Above 10	26
Above 20	21
Above 30	14
Above 40	10
Above 50	0

(*b*) Draw a cumulative frequency curve for the following :

Marks	Number of students
0—4	4
4—8	6
8—12	10
12—16	8
16—20	4

7. (*a*) Factorise x^6-64.

(*b*) Divide polynomial $y^5-2y^4+4y^3-y^2-2y+6$ by y^3-y^2+2.

8. (*a*) Evaluate $\dfrac{(3{\cdot}29)^{1/2}\times 56{\cdot}81\times 1{\cdot}328}{42{\cdot}59\times 8{\cdot}725}$

(*b*) Simplify : $\left(\dfrac{x^l}{x^m}\right)^{l^2+m^2+lm}\times\left(\dfrac{x^m}{x^n}\right)^{m^2+n^2+mn}\times\left(\dfrac{x^n}{x^l}\right)^{n^2+l^2+nl}.$

TEST PAPER—IV

1. (*a*) In a triangle, the three altitudes pass through the same point which is called......... of the triangle

(*b*) If two lines are perpendicular to the same line, they are ...... to each other.

(*c*) The decimal representation of π is non-terminating and..........

(*d*) The H.C.F. of $(x-1)(x+2)^2$ and $(x-1)^3(x+2)$ is..........

(*e*) Lower limit of the class-interval 26—33 is.........

(*f*) The intersection of any set A with the empty set is always the ....

(*g*) If $x=\sin 30°+\cos 60°$, then x equals ....

2. (*a*) If two sides of a triangle are unequal, then prove that the longer side has the greater angle opposite to it.

(*b*) In a triangle *ABC*, *D* is the mid-point of *BC* and *E* is the mid-point of *AD*. Prove that ar $(\triangle BDE)=\frac{1}{4}$ ar $(\triangle ABC)$.

3. (*a*) If a pair of opposite sides of a quadrilateral are equal and parallel, then prove that the quadrilateral is a parallelogram.

(*b*) *ABCD* is a parallelogram, *E* and *F* are the mid-points of the sides *AB* and *CD* respectively. Prove that segments *CE* and *AF* trisect the diagonal *BD*.

4. (*a*) Express $\sqrt[3]{256}$ as a mixed surd in its simplest form

(*b*) If $p(x)=x^4+2x^3-3x^2+x-1$, find the remainder when $p(x)$ is divided by $x-2$.

(*c*) A number consists of two digits whose sum is 9. If 27 is added to the number, the digits change their places. Find the number.

5. (*a*) If $\tan A=\dfrac{1}{\sqrt{3}}$ and $\tan B=\sqrt{3}$, show that

$$\sin A\cos B+\cos A\sin B=1$$

(*b*) A tall tree stands vertically on a bank of a river. At the point on the other bank directly opposite the tree, the angle of elevation of the top of the tree is 60°. At a point 20 metres behind this point on the same bank, the angle of elevation of the top of the tree is 30°. Find the height of the tree and the width of the river.

6. (*a*) Thirty-five girls of the same age were examined to record their pulse rate. The following figures were obtained while recording the number of beats per minute. Taking 55—60 (60 not included) as one of the class-intervals, construct a frequency distribution table with class intervals of equal lengths :

61, 56, 72, 65, 70, 59, 68, 82, 60, 66, 52, 59,
73, 78, 66, 61, 51, 59, 84, 68, 79, 75, 68, 58
71, 62, 57, 53, 74, 70, 74, 66, 60, 71, 54.

(*b*) Following is the distribution of ages (in years) of 32 teachers in a primary school :

Age in years	Number of teachers
25—31	10
31—37	13
37—43	5
43—49	3
49—55	1
Total	32

7. (*a*) Divide 116 among four parts such that 5 added to the first, 4 subtracted from the second, third multiplied by 3 and fourth divided by 2 give the same result.

(*b*) Find the L.C.M. of the polynomials x^3+27 and x^3-2x^2+6x+9.

8. (*a*) Simplify : $\dfrac{\left(p+\frac{1}{q}\right)^m\left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m\left(q-\frac{1}{p}\right)^n}$.

(*b*) Evaluate $\dfrac{(78{\cdot}41)\,.\,\sqrt[3]{142{\cdot}3}}{\sqrt[4]{{\cdot}1562}}$, using logarithm tables.

□ □

ANSWERS

Exercise—1 (*a*)

1. Sets (*b*), (*c*) and (*d*) are well-defined sets.
2. (*a*) $\{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
   (*b*) $\{1, 2, 3, 4, 6, 8, 12, 24\}$ (*c*) $\{2, 3, 5, 7\}$
   (*d*) $\{5, 10, 15, 20, 25, 30, \ldots\}$
3. (*a*) $P=\{2\}$ (*b*) $S=\{2, 3\}$ (*c*) $T=\{5, 6, 7, 8, 9, 10\}$
   (*d*) $M=\{1, 4, 9, 16, 25, 36, 49, 64, 81\}$
   (*e*) $A=\{\ldots\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$
4. (*a*) True (*b*) False
5. (*i*) True (*ii*) False
6. (*a*) $\{\quad\}$ (*b*) $\{1, 2, 3, 4, 5\}$
7. $\{x : x \in N, 7 < x \leqslant 21\}$
8. (*a*) $X=\{x : x$ is a prime number$\}$
   (*b*) $E=\{x : x$ is an even number and $x \leqslant 20\}$
   (*c*) $J=\{x \mid x \in I$ and $-3 \leqslant x \leqslant 3\}$
   (*d*) $K=\{x \mid x \in W$ and $12 \leqslant x < 21\}$
9. $\{x : x=2n, n \in N\}$ 10. $\{0, 1\}$ 11. $\{1, 3, 5, 9, 15, 45\}$

Exercise—1 (*b*)

1. (*a*) Finite (*b*) infinite (*c*) infinite (*d*) finite
   (*e*) infinite (*f*) infinite (*g*) infinite (*h*) finite
2. (*a*) Not equal (*b*) equal (*c*) equal (*d*) equal
3. (*a*) Equivalent (*b*) Not equivalent (*c*) Equivalent
4. (*a*) Equal (*b*) Not equal (*c*) Not equal
5. No 6. None 7. $A \neq B$ 8. $\{0\}$ and $\{\phi\}$

Exercise—1 (*c*)

1. (*a*) Non-Empty (*b*) Non-empty (*c*) Empty (*d*) Empty
   (*e*) Empty (*f*) Empty
2. (*a*) Yes (*b*) Yes
3. $\{\quad\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}$
4. $\phi, \{0\}, \{1\}, \{0, 1\}$
5. (*a*) True (*b*) True (*c*) False 6. 16 7. Yes, equal
8. (*a*) Non-disjoint (*b*) disjoint (*c*) disjoint (*d*) disjoint
9. (*a*) False (*b*) True

10. (*a*) $\{3\}$ or $\{5\}$ or $\{3, 4\}$ or $\{3, 5\}$ or $\{3, 4, 5\}$
(*b*) $\{3\}$ or $\{5\}$ or $\{3, 5\}$ (*c*) $\{3\}$ or $\{5\}$ or $\{3, 5\}$
(*d*) $\{1\}$ or $\{2\}$ or $\{1, 2\}$, etc.

Exercise—1 (*d*)

1. The set of triangles.
2. (*a*) {Letters of the English alphabet} (*b*) {natural numbers}
(*c*) The set of integers.
3. $A'=\{a, c, f\}$; $B'=\{b, c, d, e\}$; $C'=\{a, b, c, d, e, f\}$ **4.** $\{5, 6, 7, 8\}$
5. (*a*) {x : x is a composite natural number}
(*b*) {x : x is a natural number and not a perfect square}
(*c*) {x : x is a natural number not divisible by 3 and 5}
6. (*a*) $\{2, 3, 7, 17\}$ (*b*) $\{1, 2, 3, 6, 12, 17, 52\}$ (*c*) $\{2, 6, 12, 52, 56\}$
7. Set A is equal to the universal set.

Exercise—1 (*e*)

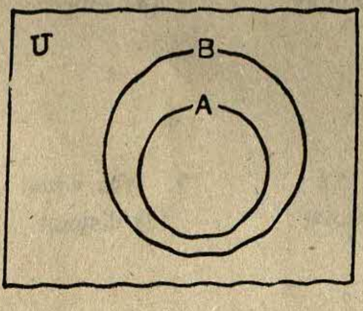
1. $A \cup B=\{a, b, c, d, e, f\}$; $A \cap B=\{b, d, f\}$
2. $C \cup D=\{p, q, r, s, x, y, z\}$; $C \cap D=\{\ \}$
3. (*a*) $\{2, 3, 4, 6, 8, 9, 10, 12, 14\}$ (*b*) $\{6, 12\}$
4. (*a*) $\{a, c, e, i, j, k, l, m, n\}$ (*b*) $\{a, k, m, n\}$
5. (*i*) $\{1, 3, 5, 7, 10, 12, 15, 18, 20, 25\}$ (*ii*) $\{10, 15\}$
6. (*a*) {x | x is a natural number} (*b*) {x | x is a real number}
(*c*) {x | x is a multiple of 3 or 5} (*d*) {x | x is an integer}
7. (*a*) $\{2\}$ (*b*) ϕ (*c*) {x | x is an integer} (*d*) {x | x is a multiple of 6}
8. $\{-5\}$
9. (*a*) $\{3, 4, 8, 12\}$ (*b*) $\{4, 12\}$ (*c*) $\{3\}$ (*d*) $\{8\}$
10. (*i*) $\{16, 17, 18, .., 25\}$ (*ii*) $\{1, 2, 3, 4, ..., 10\}$
(*iii*) $\{1, 2, 3, 4, ..., 10, 16, 17, 18, ..., 25\}$ (*iv*) ϕ
13. (*a*) $\{7, 9, 11\}$ (*b*) $\{11\}$

Exercise—1 (*f*)

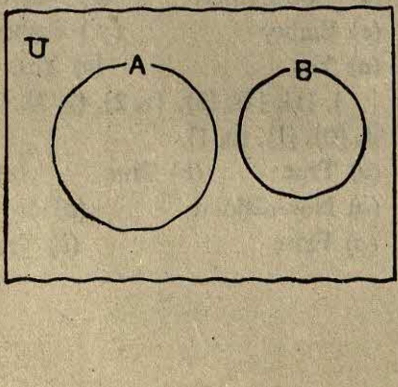
3. (*a*) $\{6, 12, 18\}$ (*b*) $\{18\}$ (*c*) $\{3, 15\}$ (*d*) $\{6, 12\}$

Exercise—1 (*g*)

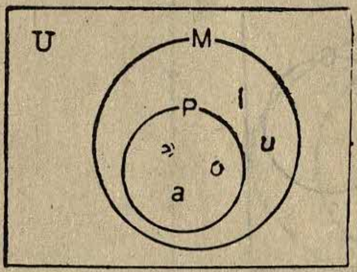
1.



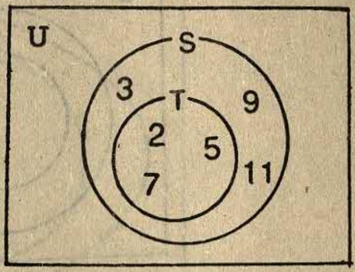
2.



3. (*a*)

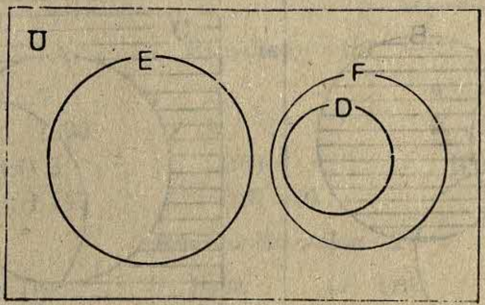


(*b*)

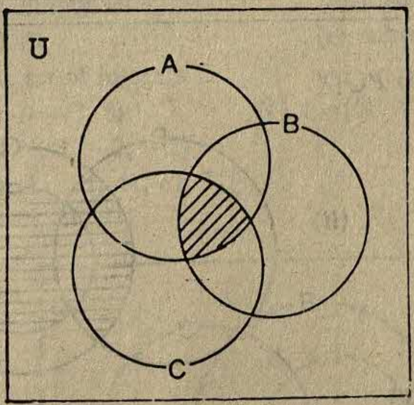


4. {1, 3, 4, 5, 7, 9, 11}

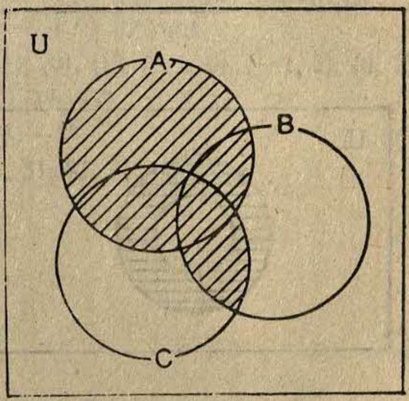
5.



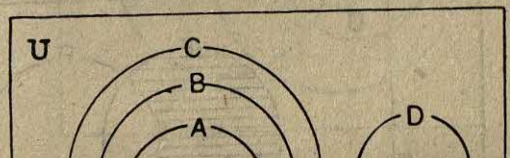
6. (*a*)



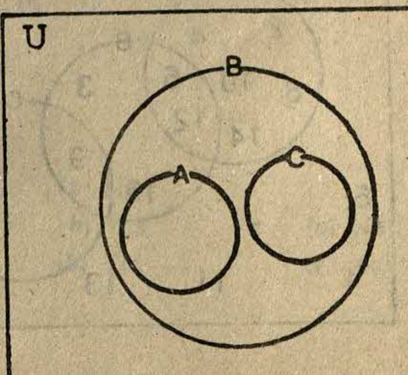
(*b*)



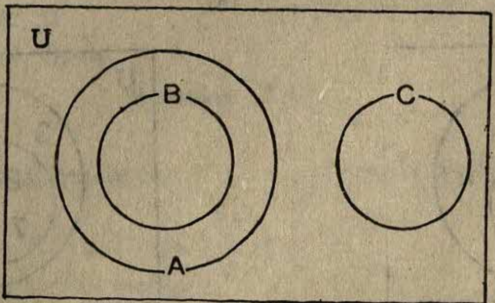
7. (*a*)



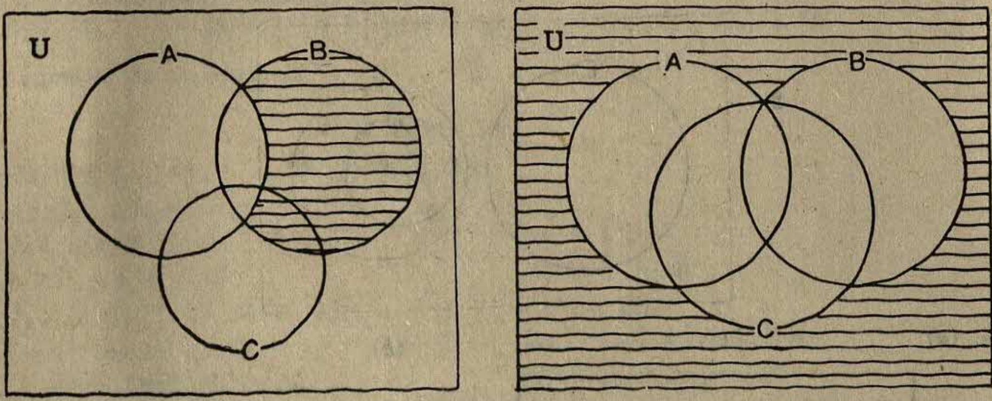
(*b*)



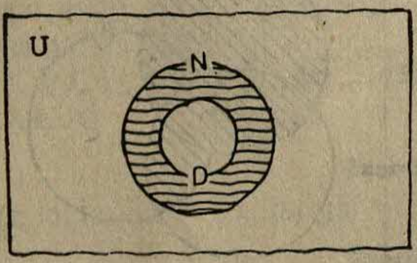
8.



9. (*a*) (*b*)

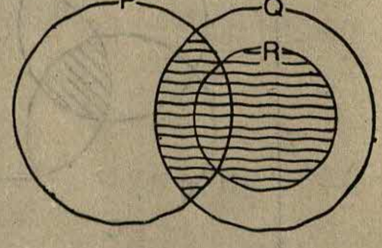


10.

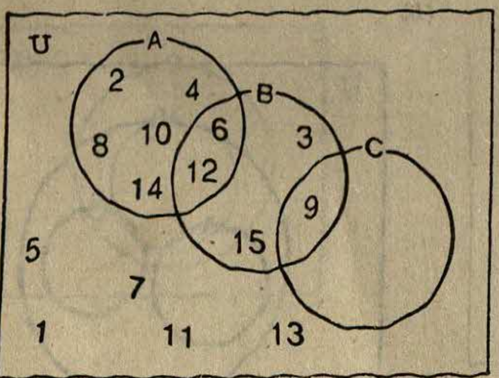


11. (*i*) $P \cup Q'$

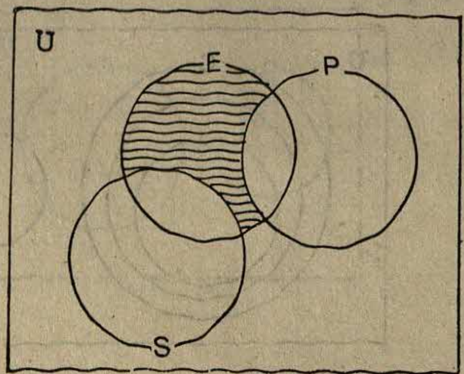
(*ii*)



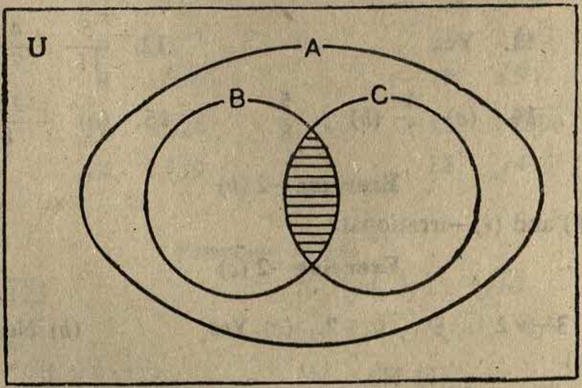
12.



13.



14.



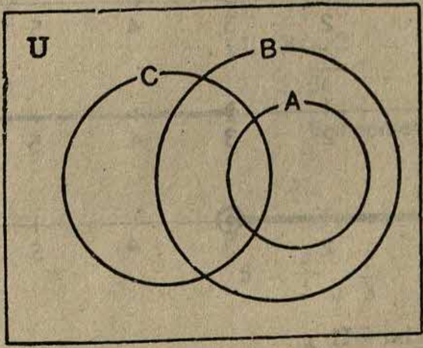
Exercise—1 (*h*)

1. 106 2. 5 3. 7 4. 17 5. 7
6. 16 7. 1200 8. 125 9. 80 10. 10
11. (*i*) 7 (*ii*) 3 (*iii*) 0 (*iv*) 4 12. 50
13. (*i*) 190 (*ii*) 140 (*iii*) 440 14. 550

Review Exercise—I

1. (*a*) False (*b*) True (*c*) False (*d*) True (*e*) True (*f*) True
2. (*a*) True (*b*) False
3. (*a*) Empty set (*b*) disjoint (*c*) $n(A)+n(B)-n(A\cap B)$
(*d*) $\cap$ (*e*) subset (*f*) disjoint
4. The set of integers 5. ϕ, $\{-1\}$, $\{0\}$, $\{1\}$, $\{-1, 0\}$, $\{-1, 1\}$, $\{0, 1\}$
7. (*a*) $D=\{3, 5\}$ (*b*) $F=\{6\}$ 8. $\{3, 5, 7, 11, 17\}$
9. (*a*) $\{-3, -2, -1\}$ (*b*) $\{-3, -1\}$
10. (*a*) $\{11, 31, 41, 61, 71\}$ (*b*) $\{21, 51, 81, 91\}$
(*c*) $\{7, 21, 91\}$ 12. U

13.



14. 8

15. 22

Exercise—2 (*a*)

1. No 2. (*a*) Yes (*b*) No
3. (*a*) Yes (*b*) Yes 4. (*a*) No (*b*) Yes (*c*) No
5. 0 6. (*a*) Yes (*b*) Yes (*c*) Yes

7. 1 ; -1 **8.** 5, -5, 1, -1 **9.** 7, 1

10. $\frac{1}{2}$ **11.** Yes **12.** $\frac{5}{11}$, $\frac{4}{7}$, $\frac{2}{3}$

13. $\frac{7}{25}$, $\frac{5}{18}$, 0·23 **14.** (*a*) $\frac{4}{7}$ (*b*) $-\frac{5}{8}$ **15.** (*a*) $-\frac{13}{8}$ (*b*) $-\frac{16}{9}$.

Exercise—2 (*b*)

1. (*a*), (*b*), (*c*), (*e*), (*f*) and (*i*)—irrational.

Exercise—2 (*c*)

1. $-4{\cdot}72$, $\sqrt{7}$, $\frac{7}{16}$, $3-\sqrt{2}$ **2.** (*a*) Yes (*b*) No

3. (*a*) Yes (*b*) No

4. (*a*) Q (*b*) R (*c*) S (*d*) R (*e*) ϕ (*f*) R

5. Yes **6.** No.

8 (*a*) $\{5, 3\}$ (*b*) $\{5, -5, 3\}$

(*c*) $\left\{5, \frac{2}{7}, -5, 3\right\}$ (*d*) $\left\{\pi, \sqrt{7}, \frac{\pi}{2}, 2+\sqrt{3}\right\}$

(*e*) $\left\{5, \frac{2}{7}, \pi, \sqrt{7}, \frac{\pi}{2}, -5, 2+\sqrt{3}, 3\right\}$

9. (*a*) The set of irrational numbers. (*b*) The set of rational numbers.

(*c*) The set of negative real numbers and zero.

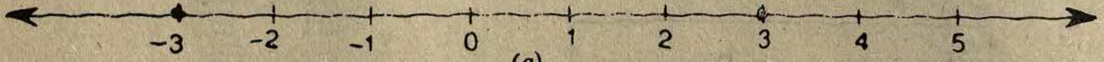
(*d*) The set of positive numbers and zero.

(*e*) The set of positive numbers. (*f*) The set of numbers not divisible by 10.

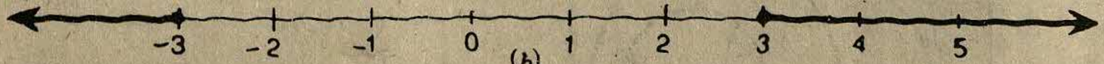
10. (*a*) 0 (*b*) 17 (*c*) $9-2\sqrt{2}$ (*d*) $10+\sqrt{3}$

11. (*a*) $x=-3, 7$ (*b*) $x=-\frac{2}{3}, 2$

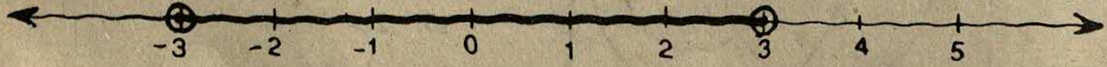
12.



(*a*)



(*b*)



(*c*)

Review Exercise—II

1. (*a*) 2 (*b*) 1, -5 (*c*) rational (*d*) even (*e*) 6

2. (*a*) 1 (*b*) 0 (*c*) -1 (*d*) 0

3. 0·375 **4.** $d=-b$ **5.** -1 **7** 6

8. $\frac{640}{99}$ **10.** $x=-3, 7$.

Exercise—3 (*a*)

1. $3\sqrt{7}$
2. $5\sqrt{5}$
3. $9\sqrt{6}$
4. $5\sqrt[3]{2}$
5. $6\sqrt[3]{3}$
6. $3\sqrt[4]{3}$
7. $\sqrt{175}$
8. $\sqrt{360}$
9. $\sqrt{847}$
10. $\sqrt{72}$
11. $\sqrt{324}$
12. $\sqrt{2500}$
13. $3\sqrt[4]{7}$
14. $2\sqrt[6]{9}$
15. $\sqrt[4]{405}$
16. $\sqrt[5]{972}$

Exercise—3 (*b*)

1. $\sqrt[12]{729}, \sqrt[12]{256}, \sqrt[12]{125}$
(*b*) $\sqrt[6]{4}, \sqrt[6]{15}, \sqrt[6]{16}$
(*c*) $\sqrt[12]{16}, \sqrt[12]{81}, \sqrt[12]{8}$
(*d*) $\sqrt[12]{27}, \sqrt[12]{81}, \sqrt[12]{4}$
2. (*a*) $\sqrt[3]{4}$ (*b*) $\sqrt[3]{6}$
(*c*) $\sqrt[5]{2}$ (*d*) $\sqrt[4]{4}$
3. (*a*) $\sqrt[4]{6}, \sqrt[3]{5}, \sqrt{3}$
(*b*) $\sqrt[6]{27}, \sqrt[3]{15}, \sqrt{21}$
(*c*) $\sqrt[6]{5}, \sqrt[3]{3}, \sqrt[4]{9}$
(*d*) $2\sqrt{5}, \sqrt[4]{1000}, 3\sqrt{4}$
4. (*a*) $\sqrt[6]{10}, \sqrt[4]{3}, \sqrt[12]{25}$
(*b*) $\sqrt[3]{6}, \sqrt[4]{10}, \sqrt{3}$
(*c*) $\sqrt[3]{2}, \sqrt[6]{3}, \sqrt[9]{4}$
(*d*) $3\sqrt{2}, 2\sqrt{3}, \sqrt[5]{7}$

Exercise—3 (*c*)

1. $5\sqrt{2}$
2. $2\sqrt{3}$
3. $11\sqrt{2}$
4. $2\sqrt{3}$
5. $7\sqrt[3]{2}$
6. $\sqrt{3}$
7. $4\sqrt{3}$
8. $7\sqrt{5}$
9. $3\sqrt{3}$
10. 0
11. 34
12. $9\sqrt[3]{4}$
13. $10\sqrt{2}$
14. $6\sqrt{5}-2\sqrt{6}$
15. $\sqrt{2}-3\sqrt{6}$
16. 0.

Exercise—3 (*d*)

1. $\sqrt{21}$
2. $\sqrt{66}$
3. $\sqrt[3]{63}$
4. $\sqrt[4]{40}$
5. $15\sqrt{21}$
6. $20\sqrt[5]{72}$
7. $\sqrt[6]{392}$
8. $\sqrt[12]{1728}$
9. $2\sqrt[6]{1125}$
10. $20\sqrt[6]{160}$
11. $3\sqrt[3]{10}$
12. $18\sqrt{6}$
13. 60
14. $84\sqrt{2}$
15. $70\sqrt[3]{2}$
16. $18\sqrt[4]{35}$
17. $\sqrt[6]{360}$
18. $2\sqrt[12]{81}$

Exercise—3 (*e*)

1. $\sqrt{2}$
2. $\sqrt[3]{4}$
3. $\sqrt[4]{\frac{5}{3}}$
4. $\sqrt[5]{\frac{7}{2}}$
5. $\frac{3}{5}\sqrt{\frac{2}{3}}$
6. $3\times\sqrt[3]{\frac{7}{10}}$
7. $\sqrt[12]{5}$
8. $\sqrt[4]{\frac{9}{5}}$
9. $\sqrt[8]{32}$
10. $\frac{5}{3}\times\sqrt[6]{\frac{9}{250}}$
11. $\frac{9}{4}\sqrt{7}$
12. $\frac{4\sqrt{5}}{9}$

Exercise—3 (*f*)

1. $2-2\sqrt{3}$
2. $3\sqrt{3}+3$
3. $3-2\sqrt{15}$
4. $5\sqrt{2}-5\sqrt{3}$
5. $30-4\sqrt{10}$
6. 6
7. $\sqrt{15}-\sqrt{10}+3\sqrt{2}-2\sqrt{3}$
8. $\sqrt{15}$
9. $54-7\sqrt{6}$
10. $5\sqrt{5}+\sqrt{15}$
11. $8\sqrt{15}+6\sqrt{10}-15\sqrt{6}-60$
12. $5\sqrt{2}+6\sqrt{5}-2\sqrt{10}-12$
13. $147-60\sqrt{6}$
14. 10
15. $49-20\sqrt{6}$.

Exercise—3 (*g*)

1. $\sqrt{3}$
2. $\sqrt{2}$
3. $\sqrt{5}$
4. $\sqrt{3}$
5. $\sqrt{5}$
6. $\sqrt{6}$
7. $\sqrt[3]{2}$
8. $\sqrt[3]{3}$
9. $\sqrt[3]{25}$
10. $\sqrt[3]{2}.\sqrt{3}$
11. $\sqrt{2}.\sqrt[3]{3}$
12. $\sqrt[3]{7}.\sqrt[3]{3}$

Exercise—3 (*h*)

1. $2\sqrt{3}$
2. $\frac{2\sqrt{7}}{7}$
3. $\sqrt{3}$
4. $\frac{4\sqrt{3}}{15}$
5. $\frac{2\sqrt{3}}{5}$
6. $\frac{4\sqrt{3}}{3}$
7. $\frac{7}{3}\sqrt{6}$
8. $\frac{5\sqrt{6}}{8}$
9. $\sqrt[3]{15}$
10. 0·894
11. 1·060
12. 2·309

Exercise—3 (*i*)

1. $\sqrt{3}-\sqrt{2}$
2. $2(\sqrt{5}+\sqrt{3})$
3. $\frac{4-\sqrt{5}}{11}$
4. $-2(\sqrt{3}+\sqrt{6})$
5. $\frac{7+5\sqrt{7}}{6}$
6. $\frac{15\sqrt{2}-6}{23}$
7. $\frac{11+6\sqrt{2}}{7}$
8. $\frac{27+10\sqrt{2}}{23}$
9. $3\sqrt{2}+2\sqrt{3}$
10. $18+13\sqrt{2}$
11. $\frac{4\sqrt{6}+\sqrt{15}}{3}$
12. $\frac{6\sqrt{10}+2\sqrt{5}-9\sqrt{2}-3}{11}$
13. 13·928
14. 98
15. 28
16. 2·063
17. $\frac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{12}$
18. $\sqrt{10}+\sqrt{5}-\sqrt{2}+1$

Review Exercise—III

1. (*a*) Fifth (*b*) pure (*c*) two (*d*) similar
   (*e*) $2\sqrt{3}+\sqrt{5}$ (*f*) $\sqrt{5}$ (*g*) conjugate (*h*) mixed
2. $\frac{34}{3}\sqrt{3}$
3. $\frac{3}{2}\sqrt{6}$
4. $\sqrt[6]{\frac{216}{625}}$

6. $-(26+15\sqrt{3})$ 7. $4-\sqrt{15}$ 8. 0

9. $\dfrac{\sqrt{30}+2\sqrt{3}-3\sqrt{2}}{4}$ 10. 1154 11. 7·872

12. 39,202.

Exercise—4 (*a*)

17. (*a*) 2nd quadrant (*b*) 3rd quadrant (*c*) 2nd quadrant
(*d*) 2nd quadrant (*e*) 3rd quadrant (*f*) Negative y-axis
(*g*) Negative x-axis (*h*) Origin (*i*) Positive x-axis

Exercise—4 (*b*)

1. The set of all non-negative real numbers. 2. (*i*) -1 (*ii*) 1
3. (*i*) $\{(1, -3), (2, -1), (3, 1), (4, 3), (5, 5)\}$ (*ii*) $\{1, 2, 3, 4, 5\}$
(*iii*) $\{-3, -1, 1, 3, 5\}$ (*iv*) $\{1, 2, 3, 4, 5\}$
4. Domain$=\{3, 6, 9\}$; Range$=\{1, 2, 3\}$
5. $f=\{(1, 1), (2, 7), (3, 17), (4, 31), (5, 49)\}$
Domain$=\{1, 2, 3, 4, 5\}$ Range$=\{1, 7, 17, 31, 49\}$
6. (*a*) $f=\{(1, 1), (\frac{1}{2}, 2), (\frac{1}{3}, 3), (\frac{1}{4}, 4), (\frac{1}{5}, 5), (\frac{1}{6}, 6)\}$.
(*b*) $f=\{(1, 2), (2, 4), (3, 6), (4, 8), (5, 10), \ldots\ldots\}$
7. (*a*) -1 (*b*) Zero (*c*) Zero (*d*) 8
8. (*a*) Zero (*b*) 6 (*c*) -2 (*d*) $\sqrt{2}-2$, 9. 49
10. (*a*) $\{2, 4, 6, 8, 10, \ldots\ldots\}$ (*b*) $\{1, 2, 3, 4, 5, \ldots\ldots\}$
(*c*) $f(2)+f(3)=f(2+3)$ (*d*) $f(2)\,f(3)\neq f(2.3)$
(*e*) $f(5x)=5f(x)$. 11. (*i*) 4 (*ii*) 144 ; No ; No
12. (*i*) $\dfrac{x}{x^2+1}$ (*ii*) $\dfrac{2x}{4x^2+1}$ (*iii*) $\dfrac{x-1}{x^2-2x+2}$
(*iv*) $-\dfrac{2}{5}$

Review Exercise—IV

1. (*a*) False (*b*) False (*c*) False
(*d*) True (*e*) True (*f*) False
3. (*a*) x independent and y dependent variables
(*b*) r independent and V dependent variables
(*c*) x independent and y dependent variables
(*d*) x independent and y dependent variables
4. 4 ; $-\dfrac{2}{3}$; -3 ; No 5. No 6. $\dfrac{3}{5}$
9. (*i*) -3 (*ii*) 0 (*iii*) 1.

Exercise—5 (*a*)

1. (*a*), (*b*), (*c*) and (*f*) are polynomials.
2. (*a*) Trinomial (*b*) Monomial (*c*) Binomial
(*d*) Trinomial (*e*) Monomial (*f*) Trinomial

3. (a) $-5-3a+6a^2+7a^3$; $7a^3+6a^2-3a-5$
(b) $-3x+6x^2+5x^3$; $5x^3+6x^2-3x$
(c) $10-5x+9x^2+8x^3+6x^4$; $6x^4+8x^3+9x^2-5x+10$
(d) $1-6m^5$; $-6m^5+1$.

4. (a) Three (b) Two (c) Four (d) Two

5. (a) $5x^{10}, 7x^{10}$; $-6x^2, -8x^2$; $7x^3, -6x^3, -5x, -9x$
(b) $-x, -4x$; $x^3, 50x^3$
(c) $-t, 5t$; $4t^3, -t^3$; $-6t^2, -8t^2$ (d) y^4, y^4.

Exercise—5 (*b*)

1. $13x$
2. $15x^2$
3. $-9x$
4. x^2
5. x^4-2x^3-x+8
6. $x^4+2x^3-3x^2+8$
7. $\frac{1}{\sqrt{2}}y^3+\frac{8}{\sqrt{10}}y^2+\sqrt{3}y-\frac{13}{10}$
8. $-\frac{13}{15}x^3-\frac{1}{\sqrt{3}}x^2+\frac{1}{\sqrt{3}}x+\sqrt{7}$
9. x^2-4x+2
10. $-x^3+x^2+4x+10$
11. Four
12. Five.

Exercise—5 (*c*)

1. $2x+3$
2. $3x-12$
3. $11x+16$
4. x^2-10x
5. $5x^3+14x$
6. $11x^4-8x^2$
7. $2x^2+25x-27$
8. $-x^2+5x-15$
9. x^3+x^2+11
10. $2x^3-2x^2+9x+5$
11. $2x^3-6x^2+3x-4$
12. Three
13. Five.
14. $\left(\frac{1}{\sqrt{2}}+\sqrt{3}+\sqrt{6}\right)x^2-\frac{5}{18}x+4$
15. $7x^3-10x^2$

Exercise—5 (*d*)

1. $6x^5$
2. $-45x^9$
3. $-12x^{12}$
4. $-60x^{10}$
5. $-\frac{1}{108}x^{10}$
6. $-x^{15}$
7. $-\frac{2}{45}x^{17}$
8. $\frac{1}{3}x^7$
9. $-\frac{\sqrt{2}}{6}x^9$

Exercise—5 (*e*)

1. $15x^4-35x^3$
2. $-15x^5+6x^3$
3. $3x^6-21x^5+24x^4$
4. $15a^5-10a^4+20a^3-15a^2$
5. $-5x^4+20x^3-45x^2$
6. $2a^6-3a^5+6a^4$
7. $10x^6-6x^5-2x^3$
8. $-10x^5+8x^4+6x^3-8x^2$
9. $9x^6-2x^4+\frac{5}{2}x^3-21x^2$
10. $x^8-\sqrt{2}x^6+\frac{1}{\sqrt{6}}x^4-\frac{5}{\sqrt{2}}x^3$

Exercise—5 (*f*)

1. $6x^2-x-15$
2. $6x^2-13x-28$
3. $20x^3-4x^2-5x+1$
4. $15x^3+20x^2+21x+28$
5. $20x^3+3x^2+23x+10$
6. $20x^3-13x^2+12x-4$
7. $9x^4-16$
8. $25x^4-81$

9. $\frac{1}{\sqrt{2}}x^3+\frac{\sqrt{2}-1}{\sqrt{6}}x^2+\frac{2}{3}x+\frac{\sqrt{2}}{\sqrt{3}}$

10. $2x^4-11x^3+37x^2-59x+56$

11. $6-\frac{1}{2}x^4-\frac{1}{6}x^6$; six

12. $\frac{3}{8}x^5-\frac{5}{6}x^3-\frac{1}{6}x^2+\frac{19}{27}x-\frac{1}{9}$; five

Exercise—5 (*g*)

1. $6x^2$ 2. $-6x^4$ 3. $\frac{-3x^3}{2}$ 4. $\frac{5}{4}$

5. $-\frac{4}{5}x^2$ 6. $\frac{5}{2}x$ 7. $\frac{\sqrt{3}}{2}x^3$ 8. $\frac{\sqrt{2}}{2}x^4$

9. $-7x^4$ 10. $-4\sqrt{3}x^3$.

Exercise—5 (*h*)

1. $2x^2-x$

2. $2x^3-5x$

3. $\frac{9}{2}x^2-3+4x$

4. $-x^3+2x^2-4x$

5. $-\frac{\sqrt{2}}{3}x^3+2x^2+3x$

6. $-\frac{8}{5}x^3-\frac{\sqrt{3}}{5}x+\frac{\sqrt{2}}{5}$

7. $4x^3-\sqrt{2}x^2+3\sqrt{3}x-\frac{\sqrt{7}}{\sqrt{3}}$

8. $-\frac{5}{2}+\frac{3}{2}x+2\sqrt{2}x^2-\frac{3}{\sqrt{2}}x^3$

Exercise—5 (*i*)

1. $x+2$
2. $2x+5$
3. $x+2$
4. $x-3$
5. $3x^2-x+3$
6. $3+3a+a^2$
7. x^2-2x+3
8. $3x^2-2x+7$
9. $x^2-3x+14$
10. Quotient$=x^3-x^2+x-1$ and remainder$=2$
11. Quotient$=2x^2+2x+6$ and remainder$=-11x-4$
12. Quotient$=25x^4-15x^2+9$ and remainder$=0$.

Exercise—5 (*k*)

1. (*a*) 1 (*b*) 1

2. (*a*) -1 (*b*) 37

3. -5

4. -140

8. 4

9. 3

10. -2

11. $a=5$, $b=24$

12. $\frac{3}{2}$

13. $p=3$, $q=-3$

14. (*a*) -4 (*b*) 4

Review Exercise—V

1. (*a*) non-negative (*b*) three (*c*) zero (*d*) zero
(*e*) zero (*f*) real numbers (*g*) $n+1$

2. (*a*) polynomial (*b*) Not a polynomial

3. $-x^5+2x^2+x-1$

4. $3x^7-x^6+3x^4-3x^2-x+3$

5. x^4+x^2+1

6. x^2+3x+3

7. Four

8. Three

9. Five
10. $1+a+a^2+a^3$
11. $3x^2+7x+2$
12. $3x^3+3x^2+7x-6$
13. $15x^5-23x^4+32x^3-24x^2+8x-3$
14. $3x^2+4x-5$
15. -18
16. $2x^4+2x^3+2x^2+5x+2$
17. $x^8-x^6+2x^2-2$
18. 10
19. Quotient $=x^2-2x-17$ and remainder $=-x+38$
20. -600
21. $p=-17$, $q=-30$

Exercise—6 (*a*)

1. $(x+7)^2$
2. $(x-9)^2$
3. $(1-3m)^2$
4. $(2p+1)^2$
5. $(2x+3)^2$
6. $(3x-7)^2$
7. $(5x-4)^2$
8. $(4x+9)^2$
9. $(2x+3y)^2$
10. $(3x-7y)^2$
11. $(4x-15y)^2$
12. $(7a-9b)^2$
13. $(2m+2n+7p)^2$
14. $4b^2$
15. $c(4ab-c^2)$
16. $5ab\,(3a-b)^2$

Exercise—6 (*b*)

1. $(8x+13y)(8x-13y)$
2. $(11a+6b)(11a-6b)$
3. $(1+9p^2q)(1-9p^2q)$
4. $(c+a+b)(c-a-b)$
5. $(4a+3b)(4a-3b)$
6. $\left(6x+\frac{1}{7}\right)\left(6x-\frac{1}{7}\right)$
7. $(3a+3b+1)(3a+3b-1)$
8. $(1+5x-5y)(1-5x+5y)$
9. $(x+1)(1-7x)$
10. $(11x+2)x$
11. $(17x+14y)(x+22y)$
12. $(4x-a-2b)(a-2b)$
13. $(7x-5y+8)(3x-5y-8)$
14. $2(3+x)(3-x)$
15. $2(1+5x)(1-5x)$
16. $2x(2x+1)(2x-1)$
17. $4y(2y+1)(2y-1)$
18. $2a(3a+2b)(3a-2b)$
19. $(4a^2+b^2)(2a+b)(2a-b)$
20. $(x^2y^2+z^2)(xy+z)(xy-z)$
21. $(x+y-z)(x-y+z)$
22. $(5x-1+6y)(5x-1-6y)$

Exercise—6 (*c*)

1. $(a+b+c)(a-b+c)$
2. $(x+y+z)(x+y-z)$
3. $(1+x+4y)(1-x-4y)$
4. $(2x+y-3z)(2x-y+3z)$
5. $(a+b-c)(a-b+c)$
6. $(3x+2a-y)(3x-2a+y)$
7. $(a+b+c-d)(a+b-c+d)$
8. $(p-q+r+1)(p-q-r-1)$
9. $(a-b+c+d)(-a+b+c+d)$
10. $(ac+bc-bd+ad)(ac-bc-bd-d)$

Exercise—6 (*d*)

1. $(x^2+x+1)(x^2-x+1)$
2. $(x^2+x-5)(x^2-x-5)$
3. $(3x^2+x-1)(3x^2-x-1)$
4. $(x^2+3xy+4y^2)(x^2-3xy+4y^2)$
5. $(2x^2+2x+1)(2x^2-2x+1)$
6. $(x^2+2xy+2y^2)(x^2-2xy+2y^2)$
7. $(2x^2+6x+9)(2x^2-6x+9)$
8. $(9x^2+12xy+8y^2)(9x^2-12xy+8y^2)$

Exercise—6 (*e*)

1. $(x+2)(x+3)$
2. $(x+3)(x+5)$
3. $(x+3)(x+4)$
4. $(x+3)(x+8)$

5. $(x+4)(x+15)$
6. $(x+7)(x+13)$
7. $(x+5)(x+14)$
8. $(x+12)(x+15)$
9. $(x+11)(x+17)$
10. $(x+13)(x+22)$

Exercise—6 (*f*)

1. $(x-2)(x-4)$
2. $(x-3)(x-5)$
3. $(x-2)(x-3)$
4. $(x-3)(x-4)$
5. $(x-5)(x-9)$
6. $(x-4)(x-21)$
7. $(x-10)(x-12)$
8. $(x-3)(x-24)$
9. $(x-11)(x-16)$
10. $(x-7)(x-32)$

Exercise—6 (*g*)

1. $(x+3)(x-2)$
2. $(x-3)(x+2)$
3. $(x+5)(x-3)$
4. $(x-5)(x+3)$
5. $(x+9)(x-5)$
6. $(x-9)(x+5)$
7. $(x+21)(x-5)$
8. $(x-12)(x+9)$
9. $(x+18)(x-7)$
10. $(x-13)(x+10)$

Exercise—6 (*h*)

1. $(2x+1)(2x+5)$
2. $(3x+2)(2x-1)$
3. $(x+3)(2x-1)$
4. $(2x+3)(x-2)$
5. $(x+4)(4x-3)$
6. $(3x-2)(2x+1)$
7. $(2x-1)(3x-5)$
8. $(3x+4)(2x-3)$
9. $(2x-3)(7x-2)$
10. $(3x+8)(4x-5)$
11. $(2x+5)(3x-1)$
12. $(x+5)(8x+3)$
13. $(x+1)(3x-7)$
14. $(3x+1)(2x-3)$
15. $(1+3y)(1-21y)$
16. $(5+x)(8-x)$

Exercise—6 (*i*)

1. $(x+5y)(x+9y)$
2. $(x+9y)(x-5y)$
3. $(x-4y)(x-28y)$
4. $(x-16y)(x+5y)$
5. $(3x+2y)(2x-5y)$
6. $(2x-3y)(4x-5y)$
7. $(3x+2y)(2x-3y)$
8. $(1+x)(3-7x)$
9. $(3x+8y)(4x-5y)$
10. $5(2x-y)(2x-5y)$
11. $(xy+4)(xy-12)$
12. $(ax+3by)(ax-5by)$
13. $(x-2)(x+5)$
14. $(4x^2+6x+9)(4x^2-6x+9)$

Exercise—6 (*j*)

1. $(x+12)(x-6)$
2. $(x+5)(x+11)$
3. $(x+8)(x-3)$
4. $(x-4)(x-5)$
5. $(x-1+\sqrt{5})(x-1-\sqrt{2})$
6. $(x-2+\sqrt{3})(x-2-\sqrt{3})$

Exercise—6 (*k*)

1. $(a+4)(a^2-4a+16)$
2. $(ab+6)(a^2b^2-6ab+36)$
3. $(4x+10)(16x^2-40x+100)$
4. $(4a+5b)(16a^2-20ab+25b^2)$
5. $(3x^2+5y^2(9x^4-15x^2y^2+25y^4)$
6. $ab(7a+4b)(49a^2-28ab+16b^2)$

7. $(2a-3b+3c)(4a^2+9b^2+9c^2-12ab-6ac+9bc)$
8. $(a-2b+1)(a^2+4b^2+1-4ab-a+2b)$
9. $(7x+y)(13x^2-4xy+19y^2)$ 10. $(3y-8x)(124x^2-138xy+39y^2)$

Exercise—6 (*l*)

1. $(3x-1)(9x^2+3x+1)$ 2. $(2-x)(4+2x+x^2)$
3. $(3a-5b)(9a^2+15ab+25b^2)$ 4. $(9x-6)(81x^2+54x+36)$
5. $(2x-10y)(4x^2+20xy+100y^2)$ 6. $(a^2-2b^2)(a^4+2a^2b^2+4b^4)$
7. $a(a^2-3b^2)(a^4+3a^2b^2+9b^4)$ 8. $(a-b)(a^2+ab+b^2)(a^6+a^3b^3+b^6)$
9. $(3x+2y-z)(9x^2+4y^2+z^2+12xy+3xz+2yz)$
10. $2(b-2a)(151a^2-466ab+364b^2)$

Exercise—6 (*m*)

1. $(x-1)(x-2)(x-3)$ 2. $(x-1)(x+2)(x-3)$
3. $(x-1)(x+2)(x-2)$ 4. $(x-1)(x-2)(x+4)$
5. $(x-2)(x-4)(x+3)$ 6. $(x+1)(x+3)(x+4)$
7. $(x+1)(x+2)(x+4)$ 8. $(x+1)(x+3)(x+5)$
9. $(x-1)(3x^2+4x+5)$ 10. $(x+1)(4x^2-x+2)$
11. $(x+2)(2x-3)(3x+2)$ 12. $(x-2)^2(2x+5)$
13. $(x-3)(x+1)(x^2+3)$ 14. $(x-2)(x-5)(x+3)(x+4)$
15. $(x-3)(x-4)(x+2)(x+5)$ 16. $(x-3)(x+1)(x+2)(2x+1)$
17. $(x-4)(x+5)(3x^2+x+2)$ 18. $(x+3)^2(x+5)(2x-5)$

Exercise—6 (*n*)

1. $x-2$ 2. $x+1$ 3. $x-1$
4. $x+2$ 5. $x-2$ 6. $x+9$
7. $x+6$ 8. $3x-4$ 9. $(x-3)^2$
10. x^3+x^2+x+1 11. $x-1$ 12. $(x+1)(x+5)$
13. $x-1$ 14. $x+1$ 15. x^2-5x+6
16. $x+3$ 17. $3x-5$ 18. $2(x^2+5x+6)$

Exercise—6 (*o*)

1. $(x+1)(x^2-1)$ 2. $(x+1)(x-2)(x+2)$
3. $(x-2)(x+2)(x^2-2x+4)$ 4. $(x^2+1)(x^2-1)$
5. $(x+4)(x+5)(x+7)$ 6. $(x-4)(x-5)(x-6)$
7. $2(x-2)(x+2)(x-3)$ 8. $6(x+2)(x+3)(x-3)$
9. $(x-3)(x^3+x^2-3x+9)$ 10. $(x-2)(x-3)(x-4)(x+3)$
11. $(x+1)(x+2)(x-4)(x-5)$ 12. $(x+1)(x+2)(x+3)(x-6)$
13. H.C.F.$=x^2+2x-1$, L.C.M.$=(3x+8)(3x^3+2x^2-11x+4)$
14. H.C.F.$=3x^2+2x+1$, L.C.M.$=(2x+7)(6x^3-11x^2-8x-5)$
15. H.C.F.$=2x^2-3x+1$, L.C.M.$=(x^2+2x-2)(2x^4-7x^3+11x^2-8x+2)$
16. $x^2-12x+35$

Review Exercise—VI

1. (*a*) True (*b*) True (*c*) False (*d*) True
2. $3a+b$ 3. $a-2$ 4. $k=-3$
5. (*a*) $x+1$ (*b*) a factor (*c*) $x-2$ (*d*) $2(x+1)(x+2)$

6. $(x-y)(2x-z)$
7. $(3a-b)(3a+b+1)$
8. $(x-y)(x+y+z)$
9. $(a+b)(a+b+1)(a+b-1)$
10. $(3+x-y)(3-x+y)$
11. $2x(8-x)$
12. $(ax+by)(bx+ay)$
13. $(x^2+1)(x+1)^2(x-1)$
14. $(a+b)(a-b+c)$
15. $(x+27)(x-4)$
16. $(x-5y)(x-24y)$
17. $(x+11)(x-50)$
18. $(5x-6y)(6x+7y)$
19. $(x-2)(x-8)$
20. $3(a-1)(3a-1)$
21. $(2a-3b)(4a^2+6ab+9b^2)$
22. $(x+3)(x-3)(x^2+3x+9)(x^2-3x+9)$
23. $(x+1)(x-3)(x^2-x+1)(x^2+3x+9)$
24. $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)(x^4-x^2y^2+y^4)$
25. $2x+3$
26. $(x+1)(x+2)(x^2+1)$
27. x^2+x-6

Exercise—7 (*a*)

1. $\{6\}$
2. $\{5\}$
3. $\{5\}$
4. $\{-5\}$
5. $\{-9\}$
6. ϕ
7. $\left\{\frac{-1}{6}\right\}$
8. $\left\{\frac{-17}{2}\right\}$
9. $\left\{\frac{-5}{4}\right\}$
10. $\{-1\}$
11. $\{4\}$
12. $\left\{\frac{8}{3}\right\}$

Exercise—7 (*b*)

1. $\{7\}$
2. $\{5\}$
3. $\{3\}$
4. $\{-4\}$
5. $\{3{\cdot}75\}$
6. $\left\{\frac{3}{4}\right\}$
7. $\{10\}$
8. $\left\{\frac{2\sqrt{3}}{3}\right\}$
9. $\{1\}$
10. $\left\{\frac{-7}{120}\right\}$
11. $\{9\}$
12. $\{2\}$
13. $\left\{\frac{3}{2}\right\}$
14. $\{1\}$
15. $x=4+4\sqrt{2}$
16. $x=\frac{-5(3+\sqrt{2})}{7}$
17. $\left\{\frac{54+9\sqrt{5}}{31}\right\}$
18. $\left\{\frac{5+\sqrt{21}}{2}\right\}$

Exercise—7 (*c*)

1. 7
2. 35, 36
3. 10
4. 16
5. 17 yrs., 68 yrs.
6. 26, 28, 30
7. 37, 39, 41
8. 60 cm
9. 60
10. 31
11. 6 yrs., 30 yrs.
12. 60 yrs., 30 yrs.
13. 27
14. 73
15. 78
16. 81
17. $\frac{21}{23}$
18. 9 km
19. Rs. 24 ; Rs. 16
20. Father 33 yrs., and Son 10 yrs.
21. 10 coins of 50 P ; 40 coins of 10 P.
22. 15

Exercise—7 (*d*)

1. 6250 sq. m. 2. Length=20 m, Breadth=12 m

3. After 3 hours at 192 km 4. Rs. 1200

5. 75 men 6. Length=12 m, Breadth=4 m

7. Length=18 m, Breadth=12 m 8. 530

9. 48 km ; 4 P.M. and 8 P.M. 10. males 8000, females 7000

11. 400 sq. m 12. Rahim Rs. 1764, Karim Rs. 1600

13. Tonga Rs. 3600, Horse Rs. 2400 14. 36 days

15. (*a*) 60 km (*b*) 6 P.M. and 10 P.M. 16. 2 km/hour

Review Exercise—VII

1. $x=7$ 2. $x=5$ 3. No

4. $x=\frac{-5}{3}$ 5. $x=-6$ 6. $x=1$

7. $x=1$ 8. Length=25 cm, Breadth=16 cm

9. Rs. 1600, Rs. 1500 10. $x=-2$ 11. $x=-3$

12. Sohrab 11 year, His father 41 years 13. 150 grams

14. Ram Rs. 6615, Shyam Rs. 6000.

Exercise—8 (*a*)

1. a^{11} 2. x^{16} 3. $8x^9$

4. $\frac{16x^8}{625y^{12}}$ 5. x 6. $a^{\frac{1}{3}}$

7. $x^{2(a+b+c)}$ 8. x^{2bc} 9. 1 10. $a^{3m+n}\,b^{2m+4n}$

11. $a^{2m-5n}\,b^{3m-4n}$ 12. $x^{m+n+2p}\,y^{m+2n+3p}\,z^{2m+3n+5p}$ 13. $x^{2bc(a+c)}$

14. $x^{ac(a+b+c)}$ 15. x^{2bc} 16. $x^{2(a^2-c^2)}$

17. x^a 18. $x^{2(a^3+b^3-c^3)}$ 19. $x^{2(a^3-c^3)}$

Exercise—8 (*b*)

1. (*a*) $\frac{1}{5}$ (*b*) 81 (*c*) $\frac{1}{27}$

(*d*) $\frac{1}{8}$ (*e*) $\frac{1}{8}$ (*f*) $1\frac{7}{9}$

2. $2\frac{1}{4}$ 3. (*a*) $\frac{1}{x^2y^2}$ (*b*) $9a^6$

4. (*a*) $\frac{3c}{b^3}$ (*b*) $5\frac{1}{3}$ (*c*) $a^{-6}\,b^3$ (*d*) $\frac{y}{x^2}$

5. $\frac{1}{50}$ 6. 1 7. $1\frac{1}{2}$ 8. 1 9. abc

10. 1 14. $x^{2(a^3+b^3+c^3)}$ 16. x^{-2a^3}

Exercise—8 (*c*)

1. (*a*) $\log_2 128=7$ (*b*) $\log_8\left(\frac{1}{4}\right)=-\frac{2}{3}$ (*c*) $\log_{5\sqrt{5}} 625=\frac{8}{3}$

2. (*a*) $6^3=216$ (*b*) $(\sqrt{5})^6=125$ (*c*) $(\cdot 5)^{-3}=8$

3. (*a*) 4 (*b*) $\frac{-3}{2}$ (*c*) $\frac{7}{4}$

4. (*a*) 4 (*b*) $-\frac{1}{6}$

5. (*a*) 8 (*b*) 4

Exercise—8 (*d*)

5. (*i*) $\frac{1}{2}(x+y)$ (*ii*) $y-x$ 6. (*i*) $3a+b$ (*iii*) $2a+2b-1$

7. $2\log x+\log y+3\log z$ 8. $3\log x+\log y+\frac{1}{2}\log z$

9. $\log x+\frac{3}{2}\log y+2\log z$ 10. $\log x+\frac{2}{3}\log y-\frac{1}{3}\log z$

14. (*a*) $\frac{x}{10}$ (*b*) y^2 (*c*) $\frac{x^2}{y}$

Exercise—8 (*e*)

1. (*a*) 7·5832 (*b*) 3·5288 (*c*) $\bar{2}$·9723 (*d*) $\bar{5}$·7882
2. (*a*) 1·5368 (*b*) 3·6769 (*c*) 2·2052 (*d*) $\bar{3}$·6087
3. (*a*) 4·3036 (*b*) $\bar{7}$·6970 (*c*) $\bar{4}$·8101 (*d*) $\bar{1}$·8201
4. (*a*) $\bar{1}$·7744 (*b*) $\bar{1}$·7809 (*c*) $\bar{1}$·9284 (*d*) $\bar{1}$·4749

Exercise—8 (*f*)

1. (*a*) 3 (*b*) 1 (*c*) $\bar{2}$
   (*d*) 0 (*e*) $\bar{3}$ (*f*) $\bar{4}$
2. (*a*) 2 (*b*) $\bar{2}$ (*c*) $\bar{4}$
3. (*a*) Five (*b*) Two (*c*) One
4. (*a*) First (*b*) Second (*c*) Third
5. (*i*) 1·5537 (*ii*) $\bar{1}$·5537 (*iii*) $\bar{4}$·5537
6. 1 7. 1 8. 1 9. 2
10. 0·3686 11. (*i*) 6443900 (*ii*) ·64439 (*iii*) ·00064439
12. (*a*) 2·10721 (*b*) $\bar{1}$·10721 13. 2·04922
14. (*a*) 1·5562 (*b*) 1·9030 15. 31
16. (*i*) 22 (*ii*) 28th

Exercise—8 (*g*)

1. (*a*) 2·6794 (*b*) 0·9708 (*c*) $\bar{1}$·8142
   (*d*) 1·9117 (*e*) $\bar{2}$·8799 (*f*) $\bar{4}$·5404
2. (*a*) 4·5660 (*b*) 0·3027 (*c*) $\bar{5}$·8939
3. (*a*) 588·9 (*b*) 57·59 (*c*) 3·468
   (*d*) 0·0578
4. (*a*) 4·785 (*b*) ·05318 (*c*) 374900
   (*d*) 0·9053 (*e*) 0·004795 (*f*) 70·06
5. 1·622 6. (*a*) 29·6 (*b*) 5·97 (*c*) 0·73

Exercise—8 (*h*)

1. 3290
2. 1·289
3. 00001068
4. 93·28
5. 3·437
6. 289·6
7. 0·5238
8. 0·3770
9. 0·06974
10. 1·498
11. 2·131
12. 9·445
13. 0·554
14. 0·0812

Exercise—8 (*i*)

1. 3·888
2. 213
3. 0·003516
4. 74330
5. 8·616
6. 17·21
7. 2·603
8. 149·7
9. 0·5948
10. 9·78
11. 1·453
12. 14·18
13. 0·1234
14. 0·1663
15. 1·167

Exercise—8 (*j*)

1. 11·48 m^3
2. 14·740 dm^2
3. 3·928 m
4. 3 years
5. 0·84 dm
6. Rs. 2188
7. Rs. 1408
8. Rs. 1,79,500
9. Rs. 6250
10. 777·4 dm^2
11. 8·75 dm
12. 123 dm^2
13. $8{\cdot}572 \times 10^7$
14. 5·1%
15. 3·4 second (approx)
16. 18·2%
17. 1327 cm^2
18. 45·16 cm
19. Rs. 54,460
20. 18,080 dm^2

Review Exercise—VIII

1. (*a*) True (*b*) True (*c*) False (*d*) False
2. $\frac{1}{1000}$
3. 411·3
4. 3
5. $\log_{10} 20$
6. 4·6738
7. $y=1$
8. $5 \log a+\frac{1}{2} \log b-\frac{2}{3} \log c$
10. log 2
11. 8
12. 9th
13. 60
14. 0·8774
15. 9·29
16. 2·032
17. 606·4
18. 386
19. 1 487
20. Rs. 397
21. Rs. 6482
22. 2·007 cm
23. 987·2 cm/sec^2
24. 2 007 dm

Exercise—9 (*a*)

1. (*a*) $\sin \theta=\frac{AB}{AC}$, $\cos \theta=\frac{BC}{AC}$, $\tan \theta=\frac{AB}{BC}$

(*b*) $\sin \phi=\frac{DE}{EF}$, $\cos \phi=\frac{DF}{EF}$, $\tan \phi=\frac{DE}{DF}$

2. (*a*) $\operatorname{cosec} \theta=\frac{MN}{LM}$, $\sec \theta=\frac{MN}{NL}$, $\cot \theta=\frac{NL}{LM}$

(*b*) $\operatorname{cosec} \phi=\frac{XY}{YZ}$, $\sec \phi=\frac{XY}{ZX}$, $\cot \phi=\frac{ZX}{YZ}$

3. $\tan QPR=\frac{QR}{PR}$, $\sec PST=\frac{PS}{PT}$, $\sin SQR=\frac{SR}{QR}$

4. (*a*) $\cos \theta$; $\sin \phi$ (*b*) $\operatorname{cosec} \theta$; $\sec \phi$ (*c*) $\tan \theta$; $\cot \phi$

Exercise—9 (*b*)

1. $\sin A=\frac{3}{5}$, $\cos C=\frac{3}{5}$, $\tan A=\frac{3}{4}$, $\cot C=\frac{3}{4}$, $\operatorname{cosec} C=\frac{5}{4}$, $\sec A=\frac{5}{4}$

2. (*i*) $\frac{5}{13}$, (*ii*) $\frac{5}{13}$, (*iii*) $\frac{12}{5}$, (*iv*) $\frac{13}{12}$, (*v*) $\frac{13}{12}$, (*vi*) $\frac{12}{5}$

3. (*a*) $\frac{5}{13}$ (*b*) $\frac{4}{3}$ **4.** $\frac{4}{5}$ **5.** (*i*) $\frac{12}{13}$ (*ii*) $\frac{5}{13}$

6. $\frac{5}{13}$; $\frac{5}{12}$ **7.** (*i*) $\frac{4}{5}$ (*ii*) $\frac{3}{4}$ **8.** $\frac{5}{13}$; $\frac{13}{12}$

9. $\frac{1}{5}$ **10.** $\frac{60}{169}$ **11.** 1 **12.** $4\frac{4}{5}$ **13.** $\frac{7}{24}$, $\frac{25}{24}$

14. $\frac{40}{9}$, $\frac{40}{41}$ **16.** $\frac{1+t^2}{1-t^2}$; $\frac{2t}{1-t^2}$ **17.** $\frac{1}{6}$

20. (*i*) $\frac{6}{5}$ (*ii*) $\frac{5}{13}$ (*iii*) $\frac{12}{13}$

Exercise—9 (*c*)

1. (*a*) True (*b*) False **2.** (*i*) 0·866 (*ii*) 3·464

3. 45°

4. 3 **5.** $\frac{\sqrt{3}}{2}$, **6.** $\frac{3}{4}$ **7.** $1\frac{1}{4}$

8. 4 **9.** 1 **15.** $\frac{1}{3}$

Exercise—9 (*d*)

1. (*a*) $\frac{1}{2}$ (*b*) $1\frac{1}{16}$ (*c*) $2\frac{1}{4}$ (*d*) 2

2. $\frac{1}{4}$ **5.** $4\frac{1}{4}$ **7.** $10\frac{3}{4}$

9. $2\frac{1}{6}$ **10.** $4\frac{1}{2}$ **11.** $-\frac{11}{16}$ **14.** $\frac{1}{2\sqrt{2}}$

Exercise—9 (*e*)

1. 15° **2.** 30° **3.** 60° **4.** 45° **5.** 45°

6. $\frac{\sqrt{3}}{2}$ **7.** ±3 **8.** (*a*) 60° (*b*) $\sqrt{3}$ (*c*) 1.

9. (*i*) 12·99 metres (*ii*) 7·5 metres **10.** 10 cm, $10\sqrt{3}$ cm

Exercise—9 (*f*)

1. $10\sqrt{3}$ metres **2.** 30° **3.** 20 metres **4.** 86·6 metres

5. $1\frac{2}{3}$ metres **6.** 27·32 metres **8.** 81·96 metres **9.** 136·6 metres

10. 692·8 m **11.** $133\frac{1}{3}$ metres **12.** 30 metres

13. 31·6 m **14.** $5\sqrt{3}$ metres ; 5 metres **15.** 26·79 metres

16. 5·464 m **17.** $25\sqrt{3}$ metres ; 25 metres from one pole.

18. 63·93 metres **19.** $\frac{h\cot\beta}{\cot\alpha}$

Review Exercise—IX

1. **(a)** False (b) False **(c)** False **(d)** True

2. **(a)** 1 (b) 0 **(c)** 1 **(d)** 45°

(e) 1 (f) 45°

3. 1 **4.** 45° **5.** 2 **7.** 1·5

8. 1·5 **9.** $\frac{1}{\sqrt{3}}$ **10.** **(i)** $\sqrt{2}$ **(ii)** 1

11. $1\frac{11}{20}$ 12. $\frac{3}{160}$ **13.** 46·19 metres **14.** 200 m

15. 34·64 m **16.** 144 m

17. **(i)** 2 cm (ii) $2\sqrt{2}$ cm **(iii)** 2·3 cm

18. **(a)** $\frac{3}{10}$ (b) 6 cos x ; 1·8 cm

Exercise—10 (a)

8.

Class Intervals	*Frequency*
0—10	4
10—20	6
20—30	3
30—40	4
40—50	6
50—60	1
Total	24

9.

Monthly rent in rupees	*Number of shops*
30—38	4
38—46	9
46—54	6
54—62	3
62—70	5
70—78	7
78—86	6
Total	40

10.

Marks	*Number of students*
1—10	6
11—20	8
21—30	11
31—40	7
41—50	8
Total	40

11.

Classes	*Frequency*
40—41	2
42—43	12
44—45	7
46—47	6
48—49	7
50—51	4
52—53	2
Total	40

12.

Percentage Marks	*Number of students*
0—9	4
10—19	3
20—29	3
30—39	10
40—49	7
50—59	9
60—69	11
70—79	5
80—89	5
90—99	3
Total	60

13.

Daily earnings in (rupees)	*Number of stores f*	*cf*
0—50	3	3
51—100	7	10
101—150	4	14
151—200	5	19
201—250	4	23
251—300	3	26
301—350	2	28
351—400	2	30

14.

Class-interval	*f*	*cf*
0—8	10	10
8—16	7	17
16—24	16	33
24—32	24	57
32—40	15	72
40—48	7	79

15.

Class-interval	*f*
0—10	5
10—20	7
20—30	20
30—40	8
40—50	5
50—60	3
Total	48

16.

Marks Class-interval	*Number of students f*
0—10	4
10—20	5
20—30	7
30—40	4
40—50	10
50—60	0
Total	30

Review Exercise—X

1. (*a*) range (*b*) 7 (*c*) 12 (*d*) 85
(*e*) frequency (*f*) class mark (*g*) primary

2.

Daily wages in rupees class-interval	*Frequency f*	*Cumulative frequency cf*
4—6	6	6
6—8	5	11
8—10	10	21
10—12	8	29
12—14	3	32
14—16	2	34
Total	34	

3.

Marks	*Frequency*	*Cumulative frequency*
3—7	3	3
7—11	10	13
11—15	12	25
15—19	8	33
19—23	5	38
23—27	9	47
Total	47	

25 students are scoring below 15 marks.

4.

Class-intervals	*Frequency*
210—230	4
230—250	4
250—270	5
270—290	3
290—310	7
310—330	5
Total	28

7. (*a*)

Marks	*Number of students*
0—10	3
10—20	5
20—30	9
30—40	3
40—50	2
Total	22

(*b*) 2 students getting more than 40 marks.

Exercise—11 (*a*)

3. (*a*) Yes (*b*) No (*c*) Yes (*d*) Yes
(*e*) No (*f*) Yes

Exercise—11 (*b*)

1. 4 cm 2. *A—C—B*
3. (*a*) 160° (*b*) 80° (*c*) 108° (*d*) 24°
4. (*a*) 80° (*b*) 55° (*c*) 13° (*d*) 37°
5. No 6. *A--B—C* 7. Yes
8. Yes 9. Yes 10. 25°
11. (*a*) 90° (*b*) 45° 12. 126°
13. 30° 14. 30° 15. 42°, 48°

Exercise—12 (*a*)

1. (*a*) 118° (*b*) 75° 2. (*a*) 103° (*b*) (180 − m)°
3. 20° 5. 100° 6. 58° 7. 16°
8. 59° 9. 47°, 44°, 89°

Exercise—12 (*b*)

1. 140°, 40°, 140°   2. 120°, 60°, 120°, 60°
4. 15°

Exercise—12 (*c*)

1. 70°
5. $\angle AGP = \angle EGH = 105°$; $\angle AGH = \angle PGE = 75°$
$\angle EFH = \angle DFN = 60°$; $\angle EFD = \angle NFH = 120°$
6. (*i*) 48°   (*ii*) 42°   13. 20°
14. $\angle CPE = 54°$, $\angle CEM = 72°$, $\angle NFD = 90°$.

Exercise—12 (*d*)

1. 35°   2. No   3. $\angle A = 78°$, $\angle ACB = 34°$
5. 36°, 54°, 90°   6. 85°, 79°, 16°   7. 36°, 60°, 84°
9. 72°   10. 18°, 54°, 108°   12. 60°
13. 30°   14. $x = 20°$, $y = 130°$

Exercise—12 (*e*)

1. 65°, 80°   2. 88°   7. 52°   8. 60°

Exercise—12 (*f*)

1. 160°   2. 123°   3. 60°, 80°, 100°, 120°
4. 36°, 72°, 108°, 144°

Exercise—12 (*g*)

1. (*a*) $180° - x°$ (*b*) $\dfrac{360}{x}$
2. (*a*) 60°   (*b*) 45°   (*c*) 36°   (*d*) 30°
3. (*a*) 108°   (*b*) 120°   (*c*) 144°   (*d*) 150°
4. (*a*) 30   (*b*) 18   (*c*) 15   (*d*) 9
5. Ten sides
6. (*a*) 10   (*b*) 12   (*c*) 18   (*d*) 20
7. Eight sides   9. 92°
10. (*a*) 8   (*b*) 11   (*c*) 15
11. Ten sides   12. 8   13. 8

Review Exercise—XI

1. (*a*) supplementary   (*b*) supplementary   (*c*) parallel
(*d*) parallel   (*e*) parallel   (*f*) 110°
(*g*) 32°   (*h*) parallel   (*i*) 75°
2. (*a*) Five   (*b*) Nine   3. 52°   5. 30°, 60°, 90°
9. (*a*) No   (*b*) Not always   (*c*) Not always

Exercise—13 (*a*)

1. $\angle B = \angle C = 40°$ 2. 65°, 65°   5. 90°, 80°
8. (*a*) 36°, 72°, 72°   (*b*) 108°, 36°, 36°   9. 20°, 80°, 80°
12. 38°, 66°   13. 104°

Exercise—13 (*d*)

4. $BC > AC$ **10.** $PA > PB > PC$

Exercise—13 (*e*)

1. (*a*) Yes (*b*) No (*c*) No (*d*) Yes

Review Exercise—XII

1. (*a*) perpendicular (*b*) greater (*c*) longer
(*d*) 40° (*e*) 50°.

Exercise—14 (*a*)

4. 135°, 45°, 135° **5.** 60°, 120°, 60°, 120°
6. $\angle CDB = 30^\circ$, $\angle ADB = 80^\circ$. **7.** 33°, 147°, 33°, 147°
8. 72°, 108°, 72°, 108°. **9.** $A = 144^\circ$, $B = 36^\circ$, $C = 144^\circ$, $D = 36^\circ$
10. 72°, 108°, 72°, 108°

Exercise—14 (*c*)

1. (*a*) Yes (*b*) No (*c*) Yes (*d*) Yes
2. Parallelograms, Rhombuses and Rectangles.
3. Rectangles, Rhombuses, Parallelograms and Squares.

Exercise—14 (*d*)

1. 3 cm **2.** 7 cm

Review Exercise—XIII

1. (*a*) True (*b*) False (*c*) False (*d*) True
(*e*) True (*f*) False (*g*) True (*h*) True
(*i*) False (*j*) False.
2. (*a*) Equal (*b*) Perpendicular (*c*) Bisect (*d*) Perpendicular
(*e*) Bisects (*f*) 5 cm (*g*) 14 cm

Exercise—15 (*a*)

1. Circumference of a circle.
2. A line parallel to the given line at a distance of 3·5 cm from it.
3. Circumference of a circle with pole as centre and of radius 10 metres.
4. They are parallel **5.** A circle
6. (*a*) Along the bisector of the angle between them *i.e.*, along diagonal.
(*b*) Along a line parallel to and midway between the two edges.
7. The sphere with *C* as centre and radius equal to the constant distance.

Exercise—15 (*b*)

1. The perpendicular bisector of the fixed line.
2. The perpendicular bisector of the line segment joining points *A* and *B*.
3. The bisectors of the angles formed by the lines *AB* and *CD*.
4. The circle with *AB* as diameter.
5. (*i*) *AC* (*ii*) *BD*
6. Right-bisector of the base.
7. On the right-bisector of the line segment joining the given points.
8. A line parallel to the level ground, at a distance=the radius of the wheel.

9. A line parallel to the given line and passing through the mid-points.
10. A line parallel to the base.
11. Arc of a circle with the centre on the junction of the wall and the ground and radius = ½ the length of the ladder.
12. Semi-circle having *AB* as diameter.
13. A line parallel to the bisector of the angle *XOY*.
14. A line perpendicular to the bisector of $\angle XOY$.

Exercise—15 (*c*)

1. (*a*) No (*b*) No
2. (*a*) Inside the triangle (*b*) On the hypotenuse (*c*) Outside the triangle
3. No 4. (*a*) 1·2 cm (*b*) 3 cm (*d*) 5·7 cm
5. Yes ; In an equilateral triangle.

Review Exercise—XIV

1. (*a*) perpendicular (*b*) equidistant (*c*) 2 : 1 (*d*) orthocentre
(*e*) same (*f*) centroid (*g*) concurrent
2. (*a*) Inside the triangle (*b*) Right-angled vertex (*c*) Outside the triangle
3. (*a*) Yes (*b*) Yes (*c*) No (*d*) No

Exercise—16(*a*)

1. (*a*) 52 cm^2 (*b*) 6 cm 3. 6·4 cm

Review Exercise—XV

1. (*a*) True (*b*) True (*c*) True
(*d*) True (*e*) False (*f*) False
2. (*a*) Parallels (*b*) Equal (*c*) Interior
(*d*) Union (*e*) Two (*f*) Altitude
(*g*) Positive (*g*) Altitude.
3. (*a*) Three (*b*) Two.

Review Exercise—XVI

1. (*a*) Basic (*b*) Angles (*c*) Distinct
(*d*) Five (*e*) Perpendicular bisector (*f*) $d \lessdot r_1 + r_2$

Exercise—18 (*a*)

1. (*a*) Two diagonals and the perpendicular bisectors of each pair of opposite sides.
(*b*) Its infinite diameters.
(*c*) Infinite lines perpendicular to it.
(*d*) Perpendicular bisectors of each pair of opposite sides.
(*e*) Its angular bisector. (*f*) Its diagonals.
(*g*) Angular bisector of the vertex. (*h*) Perpendicular bisector of parallel sides.
(*i*) Angular bisector of each of vertices.
(*j*) Its perpendicular bisector and the segment itself.
2. Four 3. Two 4. None 5. None
6. Many 7. (*a*) One (*b*) Two (*c*) Two (*d*) One
8. (*a*) Kite (*b*) Isosceles trapezium.

Review Exercise—XVII

1. (*a*) True (*b*) True (*c*) True
(*d*) False (*e*) True (*f*) False

2. (*a*) Equal (*b*) Three (*c*) Not
(*d*) Congruent (*e*) Perp. bisector (*f*) Symmetrical.

Test Paper—I

1. (*a*) 360° (*b*) Perp. to each other (*c*) Zero (*d*) Subset
(*e*) 13 (*f*) Positive (*g*) Not

4. (*a*) {1, 3, 5, 7, 9} (*b*) 5

5. (*b*) Length = 6 cm, Breadth = 3 cm.

6. (*a*) Rs. 4810 (*b*) 25·98 m.

8. (*a*)

Marks	Frequency
0—10	5
10—20	7
20—30	8
30—40	6
40—50	4

(*b*)

Class-interval	*f*	*cf*
10—20	2	2
20—30	7	9
30—40	12	21
40—50	15	36
50—60	8	44
60—70	6	50
Total	50	

Test Paper—II

1. (*a*) Equal (*b*) Bisect (*c*) Subset
(*d*) 7 (*e*) Zero (*f*) Three

4. (*a*) 1 (*b*) 5

5. (*a*) $\frac{9+4\sqrt{6}}{15}$ (*b*) $(x-6)(5x-2)$

6. (*a*) $\frac{\sqrt{3}-2}{2}$ (*b*) 109·28 m

7. (*a*) 1 (*b*) $a^4+2a^3+3a^2+4a+5$, Remainder $=a-1$

8. (*a*) 26.

Test Paper—III

1. (*a*) Greater (*b*) Bisect (*c*) Real
(*d*) Frequency (*e*) Quadratic (*f*) Disjoint
(*g*) Zero (*h*) 2

4. (*a*) The set of all real numbers ; 4, 4 (*b*) $(2x+1)(3x-2)$

5. (*a*) $\frac{4}{5}$, $\frac{3}{4}$ (*b*) 36·6 m.

6. (*a*)

Class-interval	Frequency
0—10	4
10—20	5
20—30	7
30—40	4
40—50	10
50—60	0

7. (*a*) $(x+2)(x-2)(x^2-2x+4)(x^2+2x+4)$ (*b*) y^2-y+3

8. (*a*) 0·3683 (*b*) 1.

Test Paper—IV

1. (*a*) Orthocentre (*b*) Parallel (*c*) Non-repeating
(*d*) $(x-1)(x+2)$ (*e*) 26 (*f*) Empty Set (*g*) 1

4 (*a*) $4\sqrt[3]{4}$ (*b*) 21 (*c*) 36

5. (*b*) 17·32 m, 10 m

6. (*a*)

Pulse-rate	Frequency
50—55	4
55—60	6
60—65	5
65—70	7
70—75	8
75—80	3
80—85	2

7. (*a*) 22, 31, 9, 54
(*b*) $(x+1)(x+3)(x^2-3x+9)$

8. (*a*) $\left(\frac{p}{q}\right)^{m+n}$ (*b*) $9{\cdot}149\times10^6$